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Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium and the core, and limits voting cycles

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Abstract

A close connection between the theory of competitive markets, game theory, and Arrow's impossibility theorem is given by limited arbitrage, a condition originally defined on the preferences and endowments of the traders of an Arrow–Debreu economy in Chichilnisky (1991, 1993b, 1994, 1995). This condition limits the traders' diversity, and is shown here to bound their mutually beneficial gains from trade. Limited arbitrage is necessary and sufficient for the existence of an Arrow–Debreu equilibrium (see also Chichilnisky, 1991, 1993b, 1994, 1995), and for the existence of the core (see also Chichilnisky, 1993d); it is also necessary and sufficient for the elimination of Condorcet cycles on allocations involving large utility values. Condorcet cycles are the basis for the cyclical behavior of voting rules and a building block of Arrow's impossibility theorem, so that limited arbitrage appears to be at the core of social choice theory.

JEL classification: D5

1. Introduction

The expression *limited arbitrage* is used to describe economies where only bounded, or limited, opportunities for gains are available to the traders at their initial endowments. Originally introduced in Chichilnisky (1991, 1993b, 1994, 1995), the concept is central to the problem of resource allocation and is closely associated to the social diversity of the economy (Chichilnisky, 1994). Indeed, limited arbitrage was shown to be necessary and sufficient for the existence of a competitive equilibrium in economies with or without short sales in Chichilnisky (1991, 1993b,c, 1994, 1995), and in economies with infinite dimensional spaces in Chichilnisky and Heal (1991); it was also shown to be necessary and sufficient for the existence of the core in Chichilnisky (1993d). Concise proofs are provided here. It turns out that a simple geometric interpretation can be given to limited arbitrage: here I show that it is equivalent to bounding gains from trade, namely bounding the sum of utility increases which

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the traders can achieve from reallocating their initial endowments among themselves in mutually advantageous ways (Proposition 1). From this geometry a somewhat unexpected new link emerges: a close connection with Arrow's impossibility theorem (Arrow, 1951). I establish that markets have limited arbitrage if and only if they have no Condorcet triples beyond certain utility levels (Proposition 2). This means that on choices of great importance, irrational or intransitive behavior does not arise. Since Condorcet triples are the building blocks of Arrow's theorem, limited arbitrage appears to be at the core of social choice theory. The connection between limited arbitrage and the concept of no-arbitrage used in financial markets is discussed in Section 3.

The geometry of limited arbitrage provides, therefore, a well-defined connection between three classic forms of allocation which have been considered separate and almost antagonistic until now: markets, games and social choice. The concept is fundamental for a market's operation: limited arbitrage is both necessary and sufficient for the existence of a competitive equilibrium and the core in Arrow–Debreu markets with finitely or infinitely many commodities. It is also fundamental for social choice: limited arbitrage has been shown to be equivalent to the contractability of spaces of preferences (Chichilnisky 1993c), a condition which is necessary and sufficient for the existence of social choice rules which are continuous, anonymous and respect unanimity (Chichilnisky, 1982, 1993a,b, Chichilnisky and Heal, 1983). It is somewhat surprising that while this latter set of axioms of social choice, introduced in Chichilnisky (1982), is different from Arrow's, the property of limited arbitrage is nevertheless closely connected with both sets of axioms. A formal connection between the two sets of axioms was established recently by Baryshnikov (1993).

2. Limited arbitrage and gains from trade

To offer a formal perspective one needs a few definitions.[1] An economy E has $H \geq 2$ traders who trade $N \geq 2$ commodities or assets, so that the trading space X is R^N; when short sales are not allowed, the trading space is instead R^{N+}. This paper focuses on the case of markets with short sales, but the results are quite general: for markets with as well as without short sales the reader is referred to the appendix and to Chichilnisky (1991, 1993b, 1994, 1995). A trader i is described by an initial endowment $\Omega_i \in R^N$, and by a preference represented by a utility function $u_i: R^N \to R$, $u_i(0) = 0$, which is concave and satisfies mild regularity conditions detailed in the appendix, which include all standard convex preferences used in the literature. Everything in this paper is ordinal, namely independent of the utility representation; therefore without loss of generality we may consider utilities where $\sup_{\{x: x \in R^N\}} u_i(x) = \infty$.

One wishes to identify those trading opportunities which could yield unbounded utility increases for the ith trader. These are described by net trades in the set $A_i = \{y \in R^N: \forall x \in R^N, \exists \lambda > 0: u_i(\Omega_i + \lambda y) > u_i(x)\}$, a concept originally introduced in Chichilnisky (1991, 1995), which contains global information about the trader. The trader's *global cone*, C_i, is either A_i or its closure $\bar{A}_i$.[2] The trader's *market cone* is the set of all those prices at which all

[1] See also definitions in the appendix.

[2] The *global cone* is A_i when the set of gradient directions to the indifference surfaces of u_i is closed for all i, and it is $\bar{A}_i$, the closure of A_i, when the indifferences of u_i contain no half lines for all i.

trading opportunities in i's global cone are unaffordable, $D_i = \{p \in R^N: \forall y \in C_i, \langle p, y \rangle > 0\}$. The existence of competitive equilibrium[3], of the core (Chichilnisky, 1993d), and of social choice rules has been shown to depend on the relation between the traders' market cones (Chichilnisky, 1991, 1993b, 1995; Chichilnisky and Heal, 1991); this relation also provides a framework for measuring social diversity (Chichilnisky, 1994).

Definition 1. The market economy E has limited arbitrage when all its market cones intersect: $\bigcap_{i=1}^{H} D_i \neq \emptyset$.

This means that there exists one price, the same for all traders, at which the trades they can afford only increase their utilities by limited, or bounded, amounts. The concept of limited arbitrage can also be interpreted in terms of gains from trade, defined as the maximum increment in the sum of utilities which the traders can achieve by reallocating the economy's resources among themselves:

$$\text{gains from trade} = G(E) = \sup\left(\sum_{i=1}^{H} u_i(x_i) - u_i(\Omega_i)\right),$$

where for all i, $u_i(x_i) \geq u_i(\Omega_i)$ and $\sum_{i=1}^{H} (x_i - \Omega_i) = 0$.

When $X = R^N$ and the set of gradient directions to the indifferences of u_i is closed for all i:

Proposition 1. An economy E satisfies limited arbitrage if and only if it has bounded gains from trade, namely $G(E) < \infty$.

When $\sup_{\{x: x \in R^N\}} u_i(x) < \infty$, the condition in Proposition 1 is instead: $G(E) < \sup_{x \in R^N} (\sum_{i=1}^{H} u_i(x_i) - u_i(\Omega_i))$. For a proof see the appendix.

The geometry of limited arbitrage is simple: it means that the traders' global cones cannot contain net trades which add up to zero. With two traders: $\sim\exists x_i, x_j$ such that $x_i + x_j = 0$, $x_i \in A_i$ and $x_j \in A_j$. In other words: all global cones A_i must lie on one side of a given price hyperplane.

Fig. 1 illustrates an economy E_1 with two traders and two assets which has limited arbitrage. Its global cones are A_1 and A_2 and the price line p leaves both cones on one side. Therefore net trades in directions which lead to unbounded utility gains are unaffordable by all traders from their initial endowments at price p. The gains from trade in this economy, $G(E_1)$, are bounded.

The economy of Fig. 2 does not satisfy limited arbitrage: there are two directions of net trades, $w_1' \in A_1$ and $w_1 \in A_2$, yielding unbounded increases in utility and which sum up to

[3] Limited arbitrage is the first necessary and sufficient condition for the existence of a competitive equilibrium, in markets with or without short sales and with finite (Chichilnisky 1991, 1995), or infinitely many commodities (Chichilnisky and Heal, 1991). Sufficient conditions for existence of a competitive equilibrium with short sales and infinitely many commodities were provided in Chichilnisky and Heal (1993). Previous literature is discussed in the appendix and in Chichilnisky (1991, 1995).

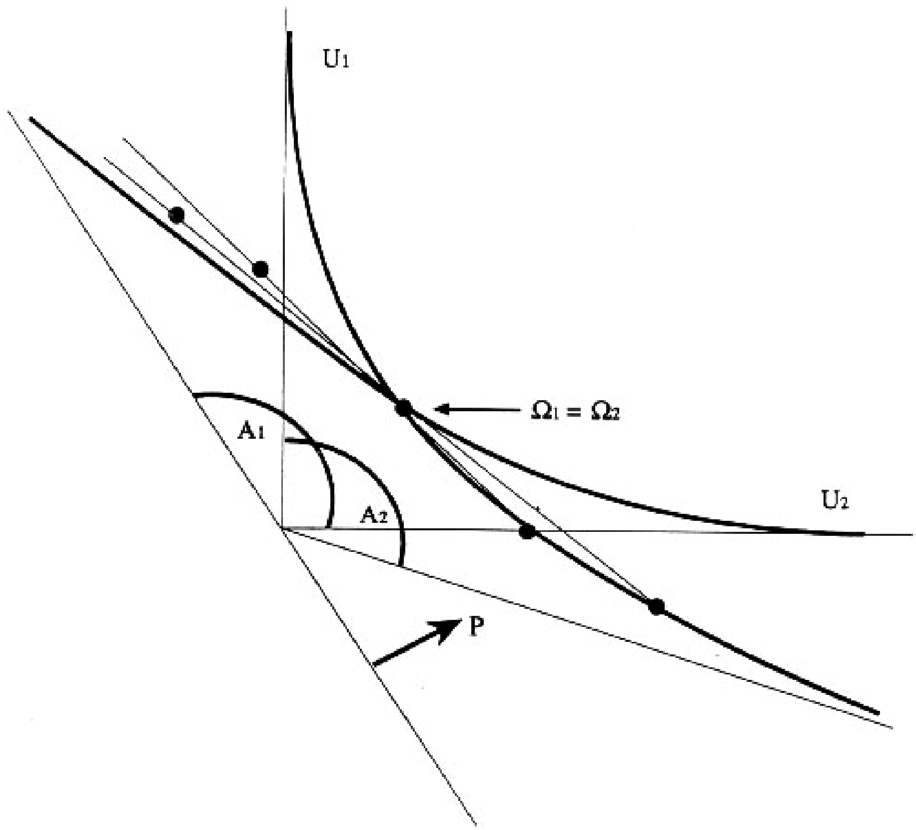

Fig. 1. Limited arbitrage is satisfied. The two global cones lie in the half-space defined by P. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment, and as shown such pairs of points lead to utility values below those of the endowments at a bounded distance from the initial endowments.

zero. Therefore, there is no price p at which all net trades in A_1 and in A_2 are unaffordable from initial endowments. The gains from trade in this economy are unbounded.

The boundedness of mutually beneficial gains from trade, which we now know from Proposition 1 to be equivalent to limited arbitrage, is fundamental to the existence of a competitive equilibrium. Formally:

Theorem 1. Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without shore sales.

For a proof see the appendix.[4] Intuitively this is reasonable: an economy such as that in Fig. 2, where traders wish to take unboundedly large and opposed trading positions, cannot reach an equilibrium. Desired trades are just too diverse to be accommodated within the same economy.

[4] This theorem does not require that preferences be increasing or smooth. The proof applies to non-satiated preferences; smoothness is only used to simplify notation (see Chichilnisky, 1991, 1993b,c, 1994, 1995).

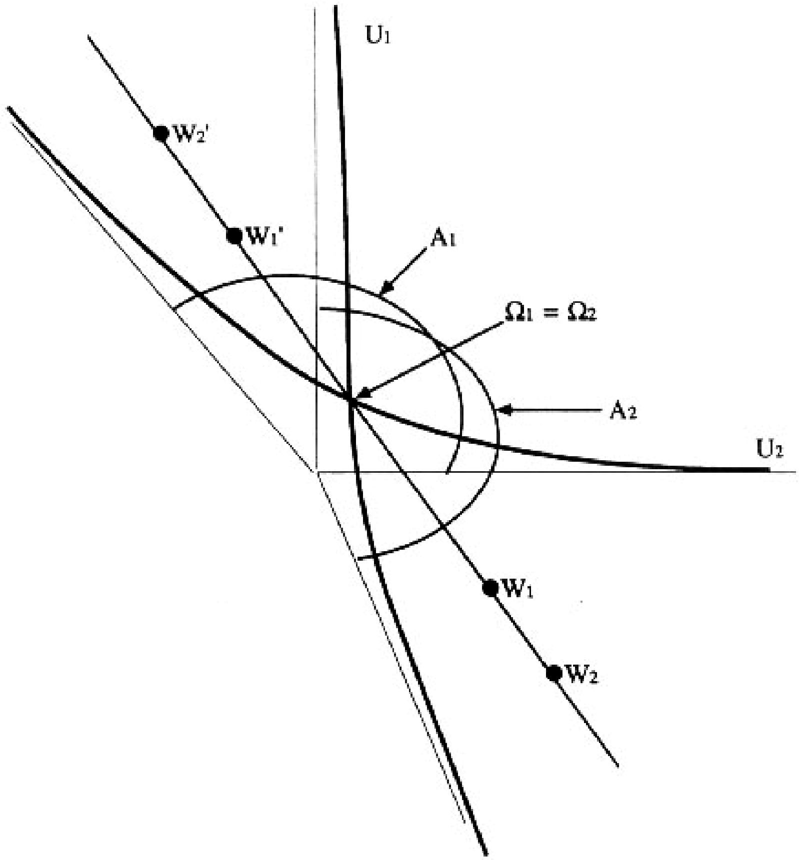

Fig. 2. Limited arbitrage does not hold. The global cones are not contained in a half-space, and there are sequences of feasible allocations such as W_1 and W_1', W_2 and W_2', which produce unbounded utilities.

The boundedness of mutually beneficial gains from trade is also fundamental for the existence of the core. Formally, when $X = R^N$:

Theorem 2. Limited arbitrage is necessary and sufficient for the non-emptiness of the core in Arrow Debreu economies.

For a proof see the appendix and Chichilnisky (1993d). The next section will relate limited arbitrage to the standard concept of no-arbitrage used in financial markets, and the following section will establish its relation with social choice theory.

3. Limited arbitrage and no-arbitrage

In financial markets an *arbitrage opportunity* exists when individuals can make unbounded gains at no cost, or, equivalently, by taking no risks. For example, buying an asset in a market where its price is low while simultaneously selling it at another where its price is high can lead to unbounded gains at no risk to the trader. *No-arbitrage* means that such opportunities do not exist, and it provides a standard way of pricing a financial asset: precisely so that no-arbitrage opportunities should arise between this and other related assets. Since trading does not cease until all arbitrage opportunities are extinguished, at a market-clearing equilibrium there is no-arbitrage.

The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy E where the traders' initial endowments are zero, $\Omega_i = 0$ for $i = 1, 2$. Here *no-arbitrage* at the initial endowments means that there are no trades which could increase the traders' utility at zero cost: gains from trade in E must be zero. By contrast, E has *limited arbitrage* when no trader can increase utility beyond a given bound at zero cost; as seen in Proposition 1, gains from trade are bounded. In summary: no-arbitrage requires that there should be *no* gains from trade at zero cost, while limited arbitrage requires that there should be only bounded or *limited* gains from trade.

The two concepts are related but nonetheless quite different. No-arbitrage is a market-clearing condition: it is used to describe an allocation at which there is no further reason to trade. It can be applied at the initial allocations, but then it means that there is no trade: the economy is autarchic and therefore not very interesting. By contrast, limited arbitrage is applied only to the economy's initial data, the traders' endowments and preferences, and it does not imply that the economy is autarchic. Quite the contrary, it is valuable in predicting whether the economy can ever reach a competitive equilibrium, and allows us to do this simply by examining the economy's initial conditions.[5] This is the subject of the next section.

4. Limited arbitrage and Arrow's theorem

I shall show next that the traders' preferences in the economy E satisfy limited arbitrage if and only if they contain no *Condorcet triples* of large utility values. Condorcet triples are building blocks of Arrow's impossibility theorem, and are at the root of the social choice problem. Thus limited arbitrage eliminates the source of Arrow's impossibility theorem for choices of large utility values.

Definition 2. A Condorcet triple is a collection of three preferences over a choice set X, represented by utilities $u_i : X \to R$, $i = 1, 2, 3$, and three choices α, β, γ within a feasible set $Y \subset X$ such that $u_1(\alpha) > u_1(\beta) > u_1(\gamma)$, $u_2(\gamma) > u_2(\alpha) > u_2(\beta)$, and $u_3(\beta) > u_3(\gamma) > u_3(\alpha)$.

Within an economy E, the social choice problem is about the choice of allocations: choices are in $X = R^{N \times H}$. An allocation $(x_1 \ldots x_H)$ is *feasible* if $\sum_i (x_i - \Omega_i) = 0$. Preferences over allocations are induced naturally by the traders' preferences over private consumption: $u_i(x_1 \ldots x_H) \geq u_i(y_i \ldots y_H) \Leftrightarrow u_i(x_i) \geq u_i(y_i)$.

Definition 3. In an economy E a family of preferences $\{u_1 \ldots u_H\}$ has a Condorcet triple of

[5] The existence of a market equilibrium does not require that the economy should have no-arbitrage at the initial endowments.

size k if for every three preferences $u_1^k, u_2^k, u_3^k \in \{u_1 \ldots u_H\}$ there exists three feasible allocations $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_H^k) \in X \subset R^{N \times H}$; $\beta^k = (\beta_1^k, \beta_2^k, \beta_H^k)$ and $\gamma^k = (\gamma_1^k, \gamma_2^k, \gamma_H^k)$ which define a Condorcet triple, and such that each trader achieves at least a utility level k at each choice: $\min_{i=1,2,3}\{[u_i^k(\alpha_i^k), u_i^k(\beta_i^k), u_i(\gamma_i^k)]\} > k$.

The following shows that limited arbitrage eliminates Condorcet triples on matters of great importance, namely on those with utility level approaching the supremum of utilities:

Proposition 2. Let E be a market economy E with no bounds on short sales. Then E has limited arbitrage if and only if for some $k > 0$, the traders' preferences have no Condorcet triples of size larger than k.[6]

A proof is in the appendix: it relies on the fact that limited arbitrage is equivalent to bounded gains from trade, Proposition 1.

Appendix

Definitions: A market economy E is defined by its trading space and its traders $E = \{X, \Omega_i \in R^{N+}, u_i : X \to R, i = 1 \ldots H\}$, where $X = R^N$, or $X = R^{N+}$ when no short sales are allowed. As already mentioned in Section 2, this paper focuses on economies where short sales are allowed but the results are quite general; the reader is referred to Chichilnisky (1991, 1993b,c, 1994, 1995) for the study of economies with or without short sales. The traders' *preferences* $u_i : X \to R$ are continuous, concave and increasing: $x \geq y \Rightarrow u_i(x) \geq u_i(y)$ and $u_i(0) = 0$.[7] Without short sales, the trading space $X = R^{N+}$; in this case one requires that if an indifference surface of positive utility intersects the boundary of R^{N-} all indifference surfaces of higher utility do too (Chichilnisky, 1991, 1995). When the trading space $X = R^N$, which is the case covered in this paper, preferences are smooth[8] (C^2), $\exists \epsilon, K > 0 : \forall x \in R^N$, $\|Du(x)\| > \epsilon$, and $\|D^2u(x)\| < K$, and either the indifferences contain no half lines, or the directions of gradients of an indifference surface form a closed set. This includes Cobb–Douglas, CES, strictly concave and linear preferences, preferences which are partly linear, preferences with indifferences which intersect the axis, and which contain half-lines. As mentioned in Section 2, the results in this paper are ordinal so that it suffices to normalize utilities so that $\sup_{\{x:x \in R^N\}} u_i(x) = \infty$. *Global cones* C_i and *market cones* D_i were defined in Section 2. The *market cone* ∂D_i of an economy E without short sales, i.e. with trading space R^{N+}, is slightly different; as defined in Chichilnisky (1991, 1994, 1995), it is: $\partial D_i = D_i \cap S(E)$ if $S(E) \subset N$, and $\partial D_i = D_i$ otherwise, where $N = \{v \in R^N : \exists i \text{ with } \langle v, \Omega_i \rangle = 0\}$, and where $S(E)$ is the set of supports to individually rational allocations: $S(E) = \{v \in R^{N+} : \exists (x_1 \ldots x_H) \in R^{H \times N+}$ with $\Sigma(x_i - \Omega_i) =$

[6] Recall that without loss of generality we have assumed that for all i, $Sup_{\{x:x \in R^N\}} u_i(x) = \infty$.

[7] The condition that preferences be increasing can be removed at no cost, and replaced by a non-satiation condition. The main results are preserved.

[8] Smoothness is unnecessary for the results; it is used to simplify notation.

0, $u_i(x_i) \geq u_i(\Omega_i)$ for all i, and $\forall z_i \in R^{N+}$, $u_i(z_i) \geq u_i(x_i) \Rightarrow \langle v, z_i - x_i \rangle \geq 0\}$. The condition of limited arbitrage is nevertheless always the same, whether $X = R^{n+}$ or $X = R^N$: it requires the non-empty intersection of all the market cones, see Section 2. The *utility set* is $U(E) = \{U_1, \ldots, U_H \in R^{H+} : \forall i = 1 \ldots H, \exists x_1 \ldots x_H$ with $U_i = u_i(x_i) \geq u_i(\Omega_i)$ where $\Sigma_{i=1}^H (x_i - \Omega_i) \leq 0\}$. The *Pareto frontier* $P(E) = \{V = V_1 \ldots V_H \in U(E)$: $\sim \exists W_1 \ldots W_H \in U(E)$ with $W_j \geq U_j \forall j$ and for some h, $W_h > U_h\}$. A competitive equilibrium is a price $p^* \in R^N$ and an allocation $(x_1^*, \ldots, x_H^*) \in X^H : \Sigma_i (x_i^* - \Omega_i) = 0$ and $\forall i\ u_i(x_i^*) = \max(u_i(x_i))$ over the set $\{x_i : \langle p^*, x_i - \Omega_i \rangle = 0\}$. An allocation $(x_1^*, \ldots, x_H^*) \in X^H$ is in the core if $\Sigma_i (x_i^* - \Omega_i) = 0$ and $\sim \exists N \subset \{1, \ldots, H\}$ and $\{y_i\}_{i \in N} : \Sigma_{i \in N} (y_i - \Omega_i) = 0$, $\forall i \in N$, $u_i(y_i) \geq u_i(x_i^*)$ and $\exists h \in N$ s.t. $u_h(y_h) > u_h(x_h^*)$.

Proposition 3. The global cones of the preference u_i are open convex sets when the sets of gradients of all indifference surfaces are closed.

Proof. By definition the global cone is A_i. A sequence $(v^n)_{n=1,2,\ldots} \subset A_i^c$ = the complement of A_i, defines half-lines $(\Gamma^n)_{n=1,2,\ldots}$, with $\sup_{\{x: x \in \Gamma^n\}}(u_i(x)) < \infty\, \forall n$. By the assumptions on u_i, $\forall n \exists y \in \Gamma^n : \langle Du_i(y), w \rangle = 0$ if $w \in \Gamma^n$. Concavity of u_i implies that $\forall w \in \Gamma^n \langle Du_i(\lambda y), w \rangle \leq 0 \forall \lambda > 1$. Assume that on two half-lines $\Gamma^n \neq \Gamma^m$ the utility u_i is eventually constant: $\exists y^n \in \Gamma^n$ and $y^m \in \Gamma^m$ such that $\forall \lambda > 1$ $\langle Du_i(\lambda y^n), w \rangle = 0$, $\forall w \in \Gamma^n$; and $\langle Du_i(\lambda y^m), w \rangle = 0$, $\forall w \in \Gamma^m$; and $u_i(y^n) < u_i(y^m)$. Let Π be a supporting hyperplane for the preferred set of u_i at λy^m; this determines a halfspace A of $R^N : \forall q \in A$, $u_i(q) < u_i(\lambda y^m)$; note that Π contains an unbounded segment of Γ^m, and A an unbounded segment of Γ^n. Therefore $\forall K > 0$, $\exists z^K \in \Gamma^n$ and $w^K \in \Pi : \|z^K - w^K\| > K$ and $\forall K$, $u_i(z^K) = u_i(y^n)$ and $u_i(w^K) = u_i(y^m)$. Since by assumption $\exists \epsilon > 0 : \forall x$, $\|Du_i(x)\| > \epsilon$, $\forall K$ the distance between z^K and $\{w \in R^N : u_i(\omega) = u_i(y^m)\}$ is bounded: $\exists T > 0$: $\forall K$, $\|z^K - w^K\| < T$, a contradiction. The contradiction arises from assuming that u_i is eventually constant on Γ^n and Γ^m with $n \neq m$; therefore $\exists n_0 : \forall j \geq n_0\, \exists y \in \Gamma^j : \langle Du_i(\lambda y), w \rangle < 0$, $\forall w \in \Gamma^j$ and $\forall \lambda > 1$. By concavity of u_i, this implies that along the half line Γ defined by $v = \lim^n v^n$, u_i is bounded, so that $v \in A_i^c$. Thus A_i^c is closed and A_i open. Convexity is immediate. □

When $X = R^N$ and gradient sets to indifference surfaces are closed.

Proposition 1 of Section 2. Let E be an economy without bounds on short sales, and normalize utilities so that $\sup_{\{x: x \in R^N\}} u_i(x) = \infty$. The Pareto frontier of the economy E is bounded if and only if the economy satisfies limited arbitrage. In particular, the economy E has bounded gains from trade, $G(E) < \infty$, if and only if it has limited arbitrage. When $\sup_{\{x: x \in R^N\}} u_i(x) < \infty$, then E has limited arbitrage if and only if the Pareto frontier is bounded away from the vector $M = (\sup_{x \in R^N} u_1(x), \ldots, \sup_{x \in R^N} u_H(x)) \in R^H$, and thus $G(E) < \sup_{x \in R^N}(\Sigma_{i=1}^H u_i(x_i) - u_i(\Omega_i))$.

Proof. By contradiction. It suffices to consider the utilities normalized to that $\sup_{\{x: x \in R^N\}} u_i(x) = \infty$. Assume E has limited arbitrage. If $P(E)$ were *not* bounded there would exist a sequence of net trades $(z_1^j \ldots z_H^j)_{j=1,2,\ldots}$ such that $\forall j$, $\Sigma_{h=1}^H z_h^j = 0$ and $\lim_{j \to \infty}(u_h(\Omega_h + z_h^j)) \to \infty$ for some h. It suffices to consider the case where $\lim_{j\to\infty}(u_h(\Omega_h + z^j_h)) \to \infty$ for all h.[9] Consider two exhaustive and exclusive cases: Case 1 and Case 2.

Case 1. For infinitely many j's, $z^j_h \in A_h$ for all h.

Limited arbitrage requires that there exists a hyperplane that leaves all the cones A_h on one side for all h, and this contradicts the fact that $z^j_h \in A_h$ for all h and $\sum_{h=1}^{H} z^j_h = 0$. Since the contradiction arises from the assumption that $P(E)$ is unbounded, $P(E)$ must be bounded in this case.

Case 2. From some j onwards, $z^j_h \notin A_h$ for some h.

Consider the sequence $\{z^j_h/\|z^j_h\|\}_{j=1,2,\ldots} \subset S^{N-1}$, the $N-1$ sphere in R^N. Since S^{N-1} is compact, it follows that there exists a sub-sequence, also denoted $\{z^j_h/\|z^j_h\|\}_{j=1,2,\ldots}$ such that $\lim_{j\to\infty} z^j_h/\|z^j_h\| = \alpha_h \in S^{N-1}$ for all $h = 1 \ldots H$. Assume first that $\alpha_h \notin A_h$. Note that it suffices to consider utilities with indifference surfaces not bounded below, since when they are bounded below, $P(E)$ is always a bounded set. By assumption, the directions of gradients of each indifference surface define a closed set. Since we assumed that $\alpha_h \notin A_h$, it follows that $\sup_{\lambda\in R^+}(u_h(\Omega_h + \lambda\alpha_h)) < \infty$. This, together with the assumption on the utilities, $\|Du(x)\| > \in$ $\forall x$ implies that if Γ is the half-line defined by the vector α_h, either $\exists w \in \Gamma$ where the gradient $Du_h(z)$ is orthogonal to Γ, or else the utility u_h achieves a maximum at $y \in \Gamma$, and is a constant beyond y. These two alternatives are exhaustive and I will show that in both it is impossible that $\alpha_h = \lim_j z^j_h/\|z^j_h\|$ with $\lim_{j\to\infty}(u_h(\Omega_h + z^j_h)) = \infty$. If the gradient $Du_h(w)$ is orthogonal to Γ at some point w, and for $\lambda > 1$ $Du_h(\lambda w)$ projected on Γ is negative, then it is also negative in a neighborhood. Therefore for directions β sufficiently close to α_h, $\exists K > 0$ such that $\sup_{x\in\Psi}(u_h(x)) < K$: a contradiction. The second alternative is that u_i utility achieves a maximum at w and remains constant thereafter on Γ. Similar reasoning, using convexity, shows that $\lim_{j\to\infty}(u_h(\Omega_h + z^j_h)) = \infty$ cannot hold either. Since these two alternatives are exhaustive, $\alpha_h \notin A_h$ is impossible, so that $\alpha_h \in A_h$ for all h where $z^j_h \notin A_h$ from some j onwards. Therefore, in Case 2, $\forall h = 1 \ldots H$ the vectors $\alpha_h \in A^c_h$, the closure of A_h, and, for some h, $\alpha_h \in A_h$. Since the cones A_h are open by Proposition 3 in the appendix, there exist nearby vectors $\beta_1 \ldots \beta_H$ s.t. $\sum_h \beta_h = 0$ and $\beta_h \in A_h$ for all h, contradicting limited arbitrage. Limited arbitrage thus implies that $P(E)$ is bounded. The reciprocal is immediate. □

Proposition 4. Let E be an economy without bounds on short sales and $H \geq 3$ traders as in Proposition 1 of Section 2. E has Condorcet triple of all sizes if and only if it does not satisfy limited arbitrage. When $\sup_{\{x:x\in R^N\}}(u_i(x)) < \infty$, the condition is instead that E has Condorcet triples of all sizes up to $\sup_{\{x:x\in R^N\}}(u_i(x))$.

Proof. Let E have limited arbitrage. Without loss of generality assume that $\forall i, \Omega_i = 0$, and choose a utility representation s.t. $\forall i$, $\sup_{\{x:x\in R^N\}}(u_i(x)) = \infty$. For each $k > 0$, let $(\alpha^k, \beta^k, \gamma^k) \in R^{3\times N\times H}$ and $u^k_1, u^k_2, u^k_3 \subset \{u_1 \ldots u_H\}$ be a Condorcet triple of size k. For each k the three allocations are feasible $\forall k$, e.g. $\alpha^k = (\alpha^k_1, \ldots, \alpha^k_H) \in R^{N\times H}$, $\sum_{i=1}^{H}(\alpha^k_i) = 0$, and $\lim_{k\to\infty}(\min_{i=1,2,3}(u_i(\alpha^k_1), u_i(\beta^k_i), u_i(\gamma^k_i)) = \infty$. There exist therefore three traders called 1, 2,

[9] By the assumptions on preferences if $\exists(u^j_1 \ldots u^j_H)_{j=1,2\ldots}: \forall j, \sum_{h=1}^{H} u^j_h = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + u^j_h)) \to \infty$ for some h, $u_h(\Omega_h + u^j_h) \geq u_h(\Omega_h)\ \forall h$, then $\exists(z^j_1 \ldots z^j_H)_{j=1,2\ldots}: \forall j, \sum_{h=1}^{H} z^j_h = 0$ and $\lim_{j\to\infty}(u_h(\Omega_h + z^j_h)) \to \infty$ for all h.

and 3 and a corresponding sequence of allocations $(\theta^k)_{k=1,2\ldots} = (\theta_1^k, \theta_2^k, \theta_3^k)_{k=1,2\ldots}$: $\forall k$, $\Sigma_i \theta_i^k = 0$ and $\forall i = 1, 2, 3$, $\sup_{k\to\infty} u_i(\theta_i^k) = \infty$. This implies that E has unbounded gains from trade, which contradicts Proposition 1. Therefore E cannot have Condorcet triples of every size.

Conversely, if E has no limited arbitrage, for any three traders, called 1, 2, 3, with preferences u_1, u_2, u_3 there exist three vectors in R^N, $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, which are part of a feasible allocation $(a_1 \ldots a_H) \in R^{N\times H}$, $\Sigma_{i=1}^H a_i = 0$. For any integer $k > 0$, and small $\epsilon > 0$ consider the vector $\Delta = (\epsilon, \ldots, \epsilon) \in R^{N+}$ and the following three allocations: $\alpha^k = (ka_1, ka_2 - 2\Delta,\ ka_3 + 2\Delta,\ ka_4, \ldots, ka_H)$, $\beta^k = (ka_1 - \Delta,\ ka_2, ka_3 + \Delta,\ ka_4, \ldots, ka_H)$ and $\gamma^k = (ka_1 - 2\Delta, ka_2 - \Delta,\ ka_3 + 3\Delta,\ ka_4, \ldots, ka_H)$; each allocation is feasible, e.g. $ka_1 + ka_2 - 2\Delta + ka_3 + 2\Delta + ka_4 + \cdots + ka_H = k(\Sigma a_i) = 0$. For each $k > 0$ the three allocations $\alpha^k, \beta^k, \gamma^k$ and the three utilities u_1, u_2, u_3 define a Condorcet triple of size $m(k)$, with $\lim_{k\to\infty} m(k) = \infty$. □

Proof of Theorem 1. Sufficiency is in Chichilnisky (1991, 1994, 1995).[10] For necessity let $X = R^N$ and $\Omega_i = 0\,\forall i$. If p^* is an equilibrium price, then $\forall i$ if $x_i \in A_i$, $\langle p^*, x_i\rangle > 0$. Furthermore, when indifferences contain no half lines, $\forall i$ if $x_i \in \overline{A_i}$ $\langle p^*, x_i\rangle > 0$: otherwise, if $\exists i$, $x_i \in \partial A_i$ s.t. $\langle p^*, x_i\rangle = 0$, then at p^* $i'th$ demand is not well defined and p^* cannot be an equilibrium, because $u_i(y) \geq \sup_{\lambda\to\infty} u_i(\lambda x_i) \Rightarrow y \in \overline{A_i}$. A similar proof holds for $X = R^{N+}$. □

Proof of Theorem 2. Let $X = R^N$. Since a competitive equilibrium is in the core sufficiency is immediate from Theorem 1. Reciprocally: a core allocation is Pareto efficient, and is therefore a competitive equilibrium for some initial endowments. Since limited arbitrage is satisfied simultaneously at all initial endowments when $X = R^N$, Theorem 1 establishes necessity. □

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References

Arrow, K., 1951, Social choice and individual values, Cowles Foundation Monograph (Wiley, New York).

Baryshnikov, Y., 1993, Unifying impossibility theorems: A topological approach, Advances in Applied Mathematics 14(4), 404–415.

Chichilnisky, G., 1982, Social aggregation rules and continuity, Quarterly Journal of Economics, May, 337–352.

Chichilnisky, G., 1991, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, Working Paper, Columbia University, March 1991.

Chichilnisky, G., 1993a, On strategic control, Quarterly Journal of Economics, February, 285–290.

Chichilnisky, G., 1993b, Markets, arbitrage and social choice, Working Paper, Columbia University, March 1991, revised March 1993, CORE Working Paper no. 9342, 1993.

[10] For one special case, $X = R^{N+}$ and preferences with no half lines in their indifferences, Werner (1987) has a sufficient condition which he states correctly, but without proof, to be necessary.

Chichilnisky, G., 1993c, Intersecting families of sets and the topology of cones in economics, Bulletin of the American Mathematical Society 29, no. 2, 189–207.

Chichilnisky, G., 1993d, Limited arbitrage is necessary and sufficient for the existence of the core, Working Paper, Stanford University, presented at the winter meetings of the Econometric Society, December 1993.

Chichilnisky, G., 1994, Social diversity, limited arbitrage and gains from trade: A unified perspective on resource allocation, American Economic Review, May, 427–434.

Chichilnisky, G., 1995, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without short sales, Economic Theory, forthcoming.

Chichilnisky, G. and G.M. Heal, 1983, Necessary and sufficient conditions for a resolution of the social choice paradox, Journal of Economic Theory 31, no. 1, 68–87.

Chichilnisky, G. and G.M. Heal, 1991, Homology and markets: Limited arbitrage is necessary and sufficient for the existence of an equilibrium in Sobolev spaces, Working Paper, Columbia University, First Boston Working Paper Series, revised 1993.

Chichilnisky, G. and G.M. Heal, 1993, Existence of a competitive equilibrium in Sobolev spaces without bounds in short sales, Journal of Economic Theory, 59, no. 2, 364–384.

Werner, J., 1987, Arbitrage and the existence of competitive equilibrium, Econometrica 55, no. 6, 1403–1418.