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The cone condition, properness, and extremely desirable commodities*

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Summary. This note links two conditions which have generated some interest in the literature and have an important role in proving the existence of an equilibrium, the second welfare theorem and the core equivalence theorem in infinite dimensional commodity spaces: These are the *cone condition* introduced in Chichilnisky and Kalman (1980), and the *properness* condition in Mas-Colell (1986a), which were studied also in Yannelis and Zame (1986), Aliprantis et al. (1987a, b) and (1989), Aliprantis and Burkinshaw (1988), Mas-Colell (1986b), Chichilnisky and Heal (1984, 1992), and Rustichini and Yannelis (1991) among others. I establish that these two conditions are the same. Indeed, the cone condition coincides also with the assumption of an *extremely desirable commodity* used in Yannelis and Zame (1986) and Rustichini and Yannelis (1991). The motivation for studying these conditions comes from the same economic application, showing the need to bring within the scope of equilibrium theory commodity spaces whose positive orthants have empty interior, a typical situation in infinite dimensional linear spaces.

1 Introduction

Crucial properties of markets, such as the existence of a competitive equilibrium, the second welfare theorem and the core equivalence theorem, all depend on being able to support convex sets with non-zero prices.[1] Prices are continuous linear functions defined on the commodity space, and the existence of such supports is proved using the Hahn-Banach theorem. When the commodity space is infinite dimensional, a problem emerges: typical infinite dimensional spaces (L_p, $p < \infty$)

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[1] The cone condition is shown to be necessary for the core equivalence theorem in Rustichini and Yannelis (1986).

have positive orthants with empty interiors,[2] so that the Hahn-Banach theorem, which requires interiors of the convex sets, cannot be applied. This biased the literature against using these highly desirable spaces as commodity spaces for many years; they were first used in the economics literature in 1977.[3] The alternative is L_∞ or C^k spaces, which have positive orthants with non empty interiors. However, a well known problem is that their price space include "purely finitely additive measures" which cannot be represented by sequences or by functions, and lead to serious problems of economic interpretation.

A solution for this problem was found in Chichilnisky (1977) and Chichilnisky and Kalman (1980). These papers established the first necessary and sufficient conditions for the existence of supporting prices for efficient paths in optimal growth models in L_p $(1 < p < \infty)$ spaces, including Hilbert spaces. Chichilnisky's (1977) condition is the existence of L_p continuous utilities being maximized at the point to be supported, and the paper characterizes such functions.[4] Chichilnisky and Kalman (1980) introduced an equivalent condition, denoted the *cone condition* hereafter, which depends on the existence of a vector at a positive distance from the cone defined by the convex set and the point to be supported. The *cone condition* has a simple interpretation. It bounds marginal rates of substitution, defining a bundle of commodities which can only be substituted by substantial amounts of other commodities. Chichilnisky and Kalman (1980, Theorem 2.1) proved that the cone condition is necessary and sufficient for the existence of a non-zero price supporting an allocation in a convex feasible set of any Banach space, whether the convex set has an interior or not.

Subsequently, Mas-Colell's (1986, Sect. 7) used what he called "*properness*" to prove the existence of a competitive equilibrium in Banach lattices[5]; he used this condition to prove the existence of supporting prices for points in convex preferred sets which may have no interior. This is the same use that the *cone condition* was given originally (Chichilnisky and Kalman 1980, Theorem 2.1).

The purpose of this note is to study the connection between these two conditions, and to establish that the two are one and the same.[6]

[2] *Interior* refers to interior in the topology of the norm. Only l_∞ and $C^k(X)$ spaces of functions $(k \geqq 0)$ and their subspaces have positive cones with nonempty interiors. Every other Banach space has a cone with empty interior.

[3] Hilbert spaces were first introduced in the economic literature to study optimal growth models and dynamic moldes of resource allocation with infinite dimensional commodity spaces in Chichilnisky (1977), Chichilnisky and Kalman (1980), Chichilnisky (1981) and Chichilnisky (1981b).

[4] A complete characterization of L_p continuous additive functionals defined on a neighborhood of the positive cone was provided in Chichilnisky (1977).

[5] Which are also Banach spaces.

[6] Mas-Colell (1986, p. 1044) refers to the connection between the two conditions stating: "Chichilnisky-Kalman (1980) imposed a condition analogous to properness at a point"; no further details are provided. Chichilnisky and Heal (1984, 1992) explain the connection between the two conditions. Yannelis and Zame (1986, p. 86) and Rustichini and Yannelis (1991 footnote 2, p. 310) also point out the precedence of the cone condition over properness.

As a by-product we also establish that the *cone condition* is the same as the assumption of *extremely desirable commodities* for complete transitive preferences used in Yannelis and Zame (1986), and Rustichini and Yannelis (1991).

Being identical, all three conditions have therefore the same economic interpretation.

2 Definitions

Let H be an ordered Banach space, called the commodity space. Let H^+ be the closure of its positive orthant and let H^* be its dual, the space of all continuous real valued linear functions on H, called also the price space. Let Y be a non-empty, closed and convex set in H possibly with an empty interior, and $x \in Y$. A positive continuous linear function p is a *supporting price* for $x \in Y$ if $\forall y \in Y$, $p(y) \geqq p(x)$. Following Chichilnisky and Kalman (1980) define the smallest cone with vertex x containing the set Y, i.e. $C(Y,x) = \{z : z = a(y-x)+x,\ y \in Y,\ a \geqq 0\}$. The *cone condition* is satisfied at x and Y if

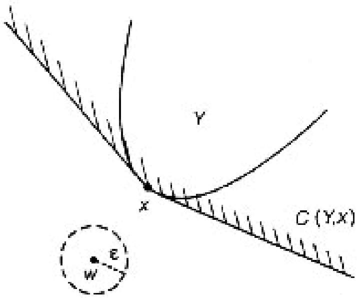

Fig. 1.

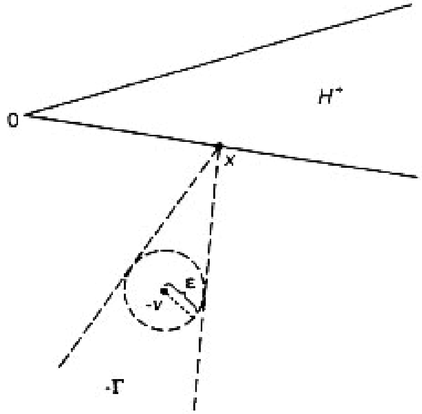

Fig. 2.

(C) There exists a vector w which is at a positive distance from the set $C(Y, x)$ (Chichilnisky and Kalman 1980, p. 25, Theorem 2.1, part a). Figure 1 illustrates.

Chichilnisky and Kalman (1980, Theorem 2.1, p. 25, part (a)) prove that (C) is necessary and sufficient for the existence of a non-zero supporting price for the convex set Y at x. Consider now a preference[7] relation $\succeq$ defined on H^+; Mas-Colell (1986, definition, p. 1043) calls the preference $\succeq$ *proper* at $x \in H^+$ if there exists some vector $v > 0$ and an open neighborhood of the origin V with the property that for any $z \in H^+$ satisfying $x - \alpha v + z \succeq x$, with $\alpha > 0$, implies $z \notin \alpha V$.[8] Equivalently, he states: "Geometrically, *properness at* x means that:

(P) There is an open cone $\Gamma \subset L$ containing a positive vector (hence Γ is not empty) such that "$(-\Gamma) \cap \{z - x \in H^+ : z \succeq x\} = \phi$."[9]

In the following we shall use (P) as the definition of properness, for which Mas-Colell (1986, p. 1043) offers the illustration in Figure 2.

Condition (P) is used by to prove the existence of a continuous positive functional p such that $p.z \geqq p.x$ whenever $z \succeq x$ (Mas-Colell 1986, p. 1043, lines 20–21), namely a supporting price. Figures 1 and 2 suggest that the conditions (C) and (P) may be the same: this intuition is formalized in what follows. Proposition 1 shows that the cone condition is identical to properness; Proposition 2 shows that it is also identical to the notion of an extremely desirable commodity.

3 Properness and the cone condition are the same

Consider the convex monotone preference $\succeq$ on H^+ defined in Sect. 2. Define the convex set $Y = \{z \in H^+ : z \succeq x\}$; Y has an empty interior if H^+ does. Let $x \in Y$. Chichilnisky and Kalman proved that condition (C) at $x \in Y$ is necessary and sufficient for the existence of a positive supporting price for Y at x. Mass-Colell states that (P) is necessary and sufficient to prove the same result (Mas-Colell 1986, p. 1043, par. 6). Since both conditions are necessary and sufficient for the existence of supporting prices, they must be equivalent. One can see this directly by showing that the two definitions are the same. In the following we prove that redefining the terms properly, for $x \in Y$, the two conditions are identical. Let $x \in Y$:

[7] A complete, monotone transitive relation $\succeq$ defined on H^+. Note that since $\succeq$ is defined on H^+, the preferred set $P_x = \{z \in H : z \succeq x\} \subset H^+$ will have an empty interior when H^+ has an empty interior. Thus the Hahn Banach's theorem cannot be invoked to prove the existence of a supporting price for the set P_x.

[8] There appears to be a misprint in Mas-Colell's definition: $z \in \varepsilon V$ should read $z \notin \varepsilon V$ instead. Furthermore, it should be pointed out that ε depends on α, and V on x. Otherwise, the statement that for $z \in L$, $z \notin \varepsilon V$ (or $z \in \varepsilon V$) is trivial, as it is always satisfied for some V and ε.

[9] It is immediate to see that the first definition of properness is the same as (P): Let u be a typical point in the open cone $-\Gamma + x$, generated by the set $((x - \alpha v) + V)$ and with vertex x, see Fig. 2. Thus $u = \mu(x - \alpha v + z) + (1 - \mu)x$, for $\mu \geqq 0$ and for $\|z\| < \varepsilon\gamma$, where γ is the norm of the open set V, $\gamma = \sup \{\beta > 0 : \|y\| \leqq \beta, \forall y \in V\}$. It is immediate that $u \notin \{z \in H^+ : z \succeq x\}$. This is because *if* $u = \mu(x - \alpha v + z) + (1 - \mu)x \succeq x$ then the first definition of properness *would imply* that $\|\mu z\| > \varepsilon\mu\gamma$, or $\|z\| > \varepsilon\gamma$, contradicting the definition of u. Therefore the first definition of properness implies the existence of an open cone $-\Gamma$ such that $-\Gamma \cap \{z - x \in H^+ : z \succeq x\} = \phi$. The converse is also immediate.

Proposition 1. *Properness* (P) *and the cone condition* (C) *are the same.*

Proof. First we show when (P) is satisfied so is (C). Let x and Y satisfy (P); then by definition there exists an open cone Γ containing a positive vector v such that $(-\Gamma)\cap\{z-x\in H^+:z\succeq x\}=\phi$. The cone $C(Y,x)=\{\lambda(z-x)+x, z\succeq x, \lambda\geqq 0\}$ cannot intersect the open set $(-\Gamma+x)=\{z\in H:z=u+x, u\in-\Gamma\}$, for if $g\in(-\Gamma+x)\cap C(Y,x)$, then $g=\lambda(z-x)+x$, so that $\lambda(z-x)\in-\Gamma$ and since $-\Gamma$ is a cone, this implies $(z-x)\in-\Gamma$, contradicting (P). Therefore $C(Y,x)\cap(-\Gamma+x)=\phi$. Since $(-v+x)\in(-\Gamma+x)$, $(-v+x)$ is at a positive distance from $C(Y,x)$, and the cone condition (C) is satisfied[10].

Conversely, assume (C). Then, by definition, there exists a vector w which can be taken to be negative when $x\geqq 0$ at a positive distance from $C(Y,x)$. Therefore there exists $\delta>0$, s.t $V_\delta=\{z\in H:\|z-w\|<\delta\}$, does not intersect $C(Y,x)$. In particular, the open cone with vertex x generated by V_δ, $(-\Gamma+x)=\{u\in H:u=(\lambda z+x), z\in V_\delta, \lambda>0\}$, does not intersect $C(Y,x)$ because $C(Y,x)$ is also a cone. Therefore, $-\Gamma$ does not intersect the set $\{z-x\in H^+:z\succeq x\}$, which is contained in $(C(Y,x)-x)$. Since $-\Gamma$ contains $w\in V_\delta$ and $(-\Gamma)\cap\{z-x\in H^+:z\succeq x\}=\phi$, condition (P) is satisfied for $x\in Y$. □

We now turn to the definition of an *extremely desirable commodity* for the convex preference $\succeq$ on H^+. Yannelis and Zame (1986) call v an extremely desirable commodity at x if $(-C)\cap\{y-x\in H^+:y\succeq x\}=\phi$, where $C=\{\alpha v-z:z\in\alpha U\}$, U an open neighborhood in H, Rustichini and Yannelis (1991, p. 319). This implies that an agent would prefer to trade any commodity z for an additional increment of the bundle v, provided that the size of z is sufficiently small compared with the increment of v. Then:

Proposition 2. *The cone condition is the same as the assumption of an extremely desirable commodity for* $\succeq$.

Proof. The definition of an extremely desirable commodity v at x is identical to condition (P) above for $\Gamma=C$, which we showed in Proposition 1 to be identical to the cone condition of Chichilnisky and Kalman (1980). □

References

Aliprantis, C.D., Brown, D.J., Burkinshaw, O.: Existence and optimality of competitive equilibrium. Berlin, New York: Springer 1989

Aliprantis, C.D., Brown, D.J., Burkinshaw, O.: Edgeworth equilibria. Econometrica **55**, 1109–1137

Aliprantis, C.D., Brown, D.J., Burkinshaw, O.: Edgeworth equilibria in production economies. J. Econ. Theory **43**, 252–291 (1987)

Aliprantis, C.D., Burkinshaw O.: The fundamental theorems of welfare economics without proper preferences. J. Mathematical Economics 17, 41–54 (1988)

Chichilnisky, G.: Non-linear functional analysis and optimal economic growth. J. Math. Anal. Appl. **61**, 504–520 (1977)

Chichilnisky, G.: Existence and characterization of optimal growth paths including models with non convexities in utilities and technologies. Rev. Econ. Stud. **48**, 51–61 (1981a)

[10] One can always choose v in (P) so that $-v+x\geqq 0$.

Chichilnisky, G.: Existence of optimal savings policies with imperfect information and non convexities. J. Math. Econ. **8**, 1–14 (1981b)

Chichilnisky, G., Kalman, P.J.: Application of functional analysis to models of efficient allocation of economic resources. J. Optimization Theory Appl. **30**, 19–32 (1980)

Chichilnisky, G., Heal, G.M.: Competitive equilibrium in L_p and Sobolev spaces. IMA Preprint Series No. 79, Institute for Mathematics and its Applications, University of Minnesota, June 1984

Chichilnisky, G., Heal, G.M.: Competitive equilibrium in Sobolev spaces without bounds on short sales. J. Econ. Theory forthcoming

Mas-Colell, A.: The equilibrium existence problem in topological vector lattices. Econometrica **54**, 1039–1053 (1986a)

Mas-Colell, A.: Valuation equilibrium and pareto optimum revisited. In: Hildenbrand, W., Mas-Colell, A. (eds.) Contributions to mathematical economics, Chapt. 17, pp. 317–331 New York: North-Holland (1986)

Rustichini, A., Yannelis, N.: Edgeworth's conjecture in economies with a continuum of agents and commodities. J. Math. Econ. **20**, 307–26 (1991)

Yannelis, N.C., Zame, W.R.: Equilibria in Banach lattices without ordered preferences. J. Math. Econ. **15**, 85–110 (1986)