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Topology and Economics: The Contribution of Stephen Smale[1]

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1. Introduction

Classical problems in economics are concerned with the solutions of several simultaneous nonlinear optimization problems, one for each consumer or producer, all facing constraints posed by the scarcity of resources. Often their interests conflict, and it is generally impossible to find a single real-valued function representing the interests of the whole of society. To deal with this problem, John Von Neumann introduced the theory of games. He also defined and established the existence of a general economic equilibrium, using topological tools [Von Neumann, 1938]. The work of Stephen Smale follows this tradition. He uses topological tools to deepen and refine the results on existence and other properties of another type of economic equilibrium, the Walrasian equilibrium (Walras [1874–77]), as formalized by Kenneth J. Arrow and Gerard Debreu [1954], and of non-cooperative equilibrium in game theory as formalized by Nash (1950). This article aims to show that topology is intrinsically necessary for the understanding of the fundamental problem of conflict resolution in economics in its various forms[2] and to situate Smale's contribution within this perspective. The study of conflicts of interests between individuals is what makes economics interesting and mathematically complex. Indeed, we now know that the space of all

[1] This is a revised version of the paper presented at Conference in Honor of Stephen Smale at the University of California Berkeley, August 1990. I thank those who commented on the presentation of this paper on August 5, 1990, and Ralph Abraham, Don Brown, Curtis Eaves, Geoffrey Heal, Morris Hirsch, Jerrold Marsden, Andreu Mas Colell and Dennis Sullivan for comments. I am particularly grateful to Kenneth J. Arrow for valuable suggestions. Hospitality and financial support from the Stanford Institute of Theoretical Economics at Stanford University is gratefully acknowledged.

[2] A similar statement is made by Von Neumann about his general equilibrium market solution: "The mathematical proof (of existence of an economic equilibrium) is possible only by means of a generalization of Brouwer's Fix-Point theorem, i.e. by the use of very fundamental *topological facts*" (Von Neumann, [1945–46], p. 1, parenthesis supplied).

individual preferences, which define the individual optimization problems, is topologically nontrivial, and that its topological complexity is responsible for the impossibility of treating several individual preferences as if they were one, i.e., aggregating them (Chichilnisky, 1980; Chichilnisky and Heal, 1983). Because it is not possible, in general, to define a single optimization problem, other solutions are sought. This article will develop three solutions, discussed below.

Because of the complexity arising from simultaneous optimization problems, economics differs from physics where many of the fundamental relations derive from a single optimization problem. The attempts to find solutions to conflicts among individual interests led to three different theories about how economies are organized and how they behave. These are *general equilibrium theory*, the *theory of games*, and *social choice theory*. Each of these theories leads naturally to mathematical problems of a topological nature. Steve Smale has contributed fruitfully to the first two theories: general equilibrium theory and the theory of games. I will argue that his work is connected also with the third approach, social choice theory, by presenting in Section 4 results which link closely, and in unexpected ways, two seemingly different problems: the existence of a general equilibrium and the resolution by social choice of the resource allocation conflict in economics (Chichilnisky, 1991).

2. A First Approach to Conflict Resolution in Economics: General Equilibrium Theory

One method of resolving social conflicts of interests is through market allocations or general equilibrium solutions. This is the area of economics to which Stephen Smale contributed most. General equilibrium theory explains how an economy behaves through a formal model of interacting markets. In particular, general equilibrium theory explains how society's resources will be allocated among the different individuals through the allocations which occur when markets clear. Such a solution is appealing because, under certain conditions, the general equilibrium allocation maximizes the Pareto order which ranks allocations among the different individuals. This is a statement of the "first theorem of welfare economics", which establishes conditions (Arrow and Hahn (1971)) under which Pareto optimality is achieved at the market clearing allocations. Pareto optimality of an allocation means that no other feasible allocation can be found at which all individuals are simultaneously better off. Pareto optimality provides a powerful rationale for studying market-induced solutions to the fundamental conflict of allocating scarce resources among several individuals.

General equilibrium theory starts with the assumption that there exist l commodities, so that a consumption bundle is described by a vector in R^l. A *pure exchange economy* is defined by three primitives, which are left unexplained:

1. Consumption sets X_i, nonempty subsets of R^l, for each agent $i = 1, \ldots, m$.
2. A vector $\mathbf{e} \in R^l$ describing a quantity of each commodity available to the economy as a whole. If the economy is *privately owned*, then there are vectors describing the initial allocations of the l commodities for each of the m-consumers or traders, $\mathbf{e}_i \in \mathbf{R}^l$, $i = 1, \ldots, m$, which describe each *individual's initial ownership of resources* or endowments, and satisfying $\sum_i e_i = e$.
3. For each trader i, a preference $\leq_i$ which is a total preorder (i.e., reflexive and transitive) on X_i.

A *pure exchange privately owned economy* is, therefore, a system, $E = \{(X_i), (\leq_i), (\mathbf{e}_i), i = 1 \ldots m\}$. The vector $\mathbf{e} = \sum_i \mathbf{e}_i \in R^l$ describes the scarcity of resources for the economy as a whole[3] and is called the initial endowment of the economy. The preference $\leq_i$ describes the criteria used by trader i to evaluate his/her allocation. These preferences are generally not linear, leading to a nonlinear optimization problem for each trader. The space of all possible preferences is an infinite-dimensional space related to, but quite different from, the space of real-valued functions on the commodity space R^l. Indeed, as it will be discussed in Section 4, the global topology of this space is responsible for the impossibility of constructing *one* preference $\leq$ which represents all of society, and which reduces the problem to a single optimization problem (Chichilnisky, 1980).

We now define in steps the general equilibrium solution. The index i describes consumers in the economy, $i = 1, \ldots, m$. Define an *attainable state s of the economy* as a list of consumption-vectors for each trader, $\{(x_i)_{i=1,\ldots,m}\}$, and a vector $\mathbf{p}$ in the positive orthant of euclidean space R^l, denoted R^{l+}, called a *price-vector*, satisfying the conditions:

(i) for every i, $x_i \in X_i$, and $\mathbf{p} \in R^{l+}$;
(ii) $\sum_i x_i \leq \sum_i \mathbf{e}_i$.

The price-vector $\mathbf{p}$ assigns to each commodity bundle z in R^l a "value" defined by the inner product $\mathbf{p} \cdot z$. For each price $\mathbf{p}$, the *budget set* $B_i(\mathbf{p})$ *of the* ith *trader* is the set $B_i(\mathbf{p}) = \{x \in R^l / \mathbf{p} \cdot x \leq \mathbf{p} \cdot \mathbf{e}_i\}$.

An attainable state s is a *general equilibrium* if, for every i, the consumption vector x_i is best according to $\leq_i$, within the ith trader's budget set $B_i(\mathbf{p})$. An attainable state s is called *Pareto optimal* if no other attainable state s' exists at which all traders are better off in terms of their preferences.

A general equilibrium, therefore, proposes an interesting solution to the resource allocation problem. Total resources are scarce, constrained as in (ii)

[3] In more general terms, production is considered as well: Smale [1974a] treated production with considerable generality; see also Debreu [1991]. However, for our purposes it will suffice to study the pure exchange economy, in which individuals trade between themselves a fixed total stock $\mathbf{e} \in R^l$ of goods.

by the initial endowment of the economy **e**. *In a general equilibrium, each trader is allocated a share* x_i *of all l commodities in such a way that each trader i maximizes his/her preference* $\leq_i$ *within their budget sets* $B_i(\mathbf{p})$*, and without exceeding society's total resources* $\sum_i x_i \leq \mathbf{e}$.

Note that auxiliary (dual) variables have been introduced: prices **p** which are vectors in R^l (one for each commodity). These variables play an important role in the definition of an equilibrium. This was already noted by Von Neumann [1938] who wrote: "Another feature of our theory, so far without explanation, is the remarkable duality (symmetry) of the monetary variables (prices y_j, interest factor β) and the technical variables." The use of prices in defining equilibria and, thus, characterizing Pareto optimal allocations is one of the fundamental insights provided by welfare economics. This has also been the subject of Smale's work (e.g., Smale [1973, 1974b, 1974d, 1975, 1976d]). The general equilibrium solution to the resource allocation problem defined by Arrow and Debreu (1954) satisfies the following remarkable properties: It respects the value of private ownership because $\sum_i \mathbf{p}_i x_i \leq \sum_i \mathbf{p}_i \mathbf{e}_i$, and it optimizes the Pareto order on social allocations. This is the main claim for its use as an acceptable form of resource allocation.

Topology has long been connected with the general equilibrium of markets. The connection between topology and general equilibrium models in economics has been established in a number of pieces starting with Von Neumann's [1938]. As its title indicates, Von Neuman's article proved a generalization of Brouwer's Fixed-Point Theorem to establish the existence of an economic equilibrium. When the Walrasian model of a general equilibrium (Walras [1874–77]) was formalized by Arrow and Debreu [1954], a fixed-point theorem was also used to establish the existence of an equilibrium. Indeed, all proofs of existence of an economic equilibrium use topological tools.

Until Smale's contribution, all proofs of existence of a general equilibrium relied on fixed-point theorems, and convexity was generally required. As Debreu [1993] points out, Smale's work coincided with the return of the differentiable tradition in economics, a tradition that had been abandoned in the 1950s and 1960s in favor of the convex approach.

Smale's papers on general equilibrium use a different topological approach. The results of M. Hirsch [1963] on the nonretractibility of a cell onto its boundary, Lemke's [1965] algorithm for affine cases, and Eaves's extension to nonlinear cases [1971, 1972], provided topological methods for computing an equilibrium and a precedent for Smale's approach to proving the existence of an economic equilibrium. Existence is established by studying the behavior of certain one-dimensional manifolds starting at the boundary of a set (the price space) and tending to a general equilibrium. Smale's proof of the existence of a general equilibrium follows this general line, using Sard's theorem, the implicit function theorem, and the classification of one-dimensional manifolds.

As in Von Neumann [1938], Smale [1974a, 1975] reduces the problem to the existence of a zero of several simultaneous nonlinear equations, each describing the behavior of one market. The zeros represent the market-clearing positions in each market. This is an old approach to the problem to which he gives a new light. He considers the standard excess demand of the economy at each price $\mathbf{p}$, denoted $z(\mathbf{p})$. This is defined as the difference between two vectors: the *total demand* vector [the sum of the optimal choices of all the traders at their budget sets $B_i(\mathbf{p})$] and the *total supply* (the social endowment $\mathbf{e}$). Excess demand $z(\mathbf{p})$ is a correspondence defined on the *price space* which is the unit simplex in R^l, $\Delta_l = \{\mathbf{p} \in R^{l+}/\sum \mathbf{p}_j = 1\}$, a closed convex subset of R^l, and with values in the commodity space R^l. Smale makes assumptions which ensure that $z(\mathbf{p})$ is a well-defined C^2 function from the price space Δ_l into R^l. For each $\mathbf{p}$, the vector $z(\mathbf{p})$ is called the "excess demand" of the economy because it is the difference between what is demanded and what is supplied of each good.

The zeros of $z(\mathbf{p})$ are the general equilibria of the economy, called the "equilibrium prices." At an equilibrium price $\mathbf{p}$ [one satisfying $z(\mathbf{p}) = 0$], the market clears and traders are within their optimal behavior. Thus, Smale's interest in finding the zeros of a function from R^m into R^m (e.g., Hirsch and Smale [1979]).

Locating the zeros of the excess demand function is achieved by studying the behavior of a one-dimensional manifold. This manifold is the inverse image of a regular value of the function

$$\varphi^*(\mathbf{p}) = \varphi(\mathbf{p})/\|\varphi(\mathbf{p})\|$$

where

$$\varphi(\mathbf{p}) = z(\mathbf{p}) - \mathbf{p}\left[\sum_h z^h(\mathbf{p})\right].$$

The zeros of $z(\mathbf{p})$ are the same as the zeros of $\varphi(\mathbf{p})$, so that φ^* is not defined on the set of equilibria, denoted $\sum$. Thus, φ^* is defined only on $\Delta_l - \sum$. Consider a regular value y of φ^*. A connected component of the one-dimensional manifold defined by the inverse image of y, starting at the boundary of Δ_l, must end outside the domain of definition of φ^*, and, thus, at a zero of $\varphi(x)$. Therefore, by following such a manifold, one finds an equilibrium (Smale, 1974a).

Smale's proof is interesting in that it can be extended in several ways to *compute* the equilibrium, by following related one-dimensional manifolds starting at the boundaries of the price space. Indeed, this became Smale's work on the Global Newton Method [1976b], related to his work with M. Hirsch "On Algorithms for solving $f(x) = 0$" (Hirsch and Smale, 1979).

Smale's work on existence of an equilibrium also provided very useful insights into the approximation of equilibrium through "nonequilibrium" paths (Smale, 1976a, 1976c, 1977, 1978). Here, Smale attacked also the problem of how the economy arrives at an equilibrium, a problem which is not yet fully solved but to which he contributed valuable technical insights. In addition, Smale explored fruitfully the question of trading outside of an equilibrium [1976c], the so called "non-tatonnement" model. This refers to a nontrivial variant of the Walrasian model which allows trading even though markets have not cleared (Arrow and Hahn, 1971).

Indeed, one of the difficulties with the Walrasian general equilibrium model is that trading takes place only at an equilibrium. An interpretation of this is that, in a Walrasian economy, there exists an "auctioneer" or "government" agent which controls the market in the sense of allowing "bets" at those prices that do not clear the market, but not allowing trading unless prices are achieved at which all markets clear. The interpretation of the auctioneer is problematic because it is at odds with the desire to view a market as a purely decentralized institution, one with no role for a "government" or other institution who knows everyone's position. Smale's work includes cases which go beyond this limitation, cases where trading takes place outside of an equilibrium. He studied conditions for the convergence of such trading to a general equilibrium [1976c].

Added to the above results, which use differentiable and algebraic topology in various forms, the Global Newton Method specified by Smale [1976b] modifies the standard Newton method (e.g., Arrow and Hahn [1971]) so that it becomes well-defined on the "singularities" where the Newton method is not, for generic sets of initial conditions. Again his approach uses topology to determine how to "continue" the Newton method through a singularity. His work on the computation of a zero of the excess demand functions led him, in turn, to develop in recent years a fruitful approach to the development of more general algorithms for the computation of zeros of a function of R^m into R^m.

2.1. Equilibrium in Nonconvex Economies

Smale also contributed to the understanding of the deterministic properties of equilibrium, i.e., to the number of equilibria of an economy. Using Sard's theorem, a technique used also by Debreu [1970], Smale [1974a] established the generic local uniqueness of equilibrium in rather general models, where production is allowed, even nonconvex production. This is a realistic assumption on the so-called "returns to scale" of the economy, which allows for "increasing" as well as "decreasing returns in production." The convexity of production sets derives from the assumption that added units of inputs produce proportionally fewer output. Instead, *increasing* returns means that production becomes *more efficient* when the quantity produced increases: An added unit of input produces proportionally a higher amount of output. With increasing returns, production sets are not convex. Economic production with informational inputs (such as, for example, those involving communications) typically exhibit increasing returns to scale, as do all industries with large initial fixed costs of production (such as R&D). Indeed, some of the most interesting and productive new technologies have this property.

Smale established properties of such nonconvex economies with simple methods, in an area which subsequently became important in economics in terms of its ability to yield new insights on the welfare properties of equilibria. With nonconvex technologies, a general equilibrium reinterpreted as Smale does, in differentiable terms, becomes essentially what is usually called a *marginal cost pricing equilibrium.*[4] Smale does not call attention to this fact, which has largely been overlooked in the literature.

Smale's framework [1974a] is as follows. He works with variable endowment-vectors $\mathbf{e}_i$ for traders, $i = 1, \ldots, m$. Each individual has a fixed consumption set denoted $P \subset R^l$ (previously X_i). The economy is thus defined by one vector $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_m) \in P^m$, and by allocations of total resources, namely, of $\sum_i \mathbf{e}_i$, to the consumers denoted $x = (x_1, \ldots, x_m)$ such that $\sum_i x_i = \sum_i \mathbf{e}_i$. A trader's preference is defined by a C^2 real-valued function u_i defined on P, with no critical points, and thus having a well-defined gradient vector $g_i(x_i)$, $\forall x_i \in P$. Smale's definition of the set Σ of equilibria is then

$$\Sigma = \left\{(\mathbf{e}, x) \in P^m \times P^m / \forall i,\ g_i(x_i) = \mathbf{p} \text{ and } \mathbf{p} \cdot x_i = \mathbf{p} \cdot \mathbf{e}_i, \sum_i x_i = \sum_i \mathbf{e}_i\right\}.$$

Smale's definition corresponds to the previously given definition of a general equilibrium (which is, in turn, in Debreu's [1993]) when preferences are concave and the trader's endowments $\mathbf{e}_i$ are fixed.

Smale replaces the optimization of preferences by a "common gradient condition": $\forall i,\ g_i(x_i) = \mathbf{p}$. This condition is equivalent to the condition of optimizing traders' preferences on their budget sets *when preferences are concave*, but not otherwise.

Smale's definition of a general equilibrium is extended to a similar definition with production, in which case it implies optimization (of profits by producers) only when there is convexity (of production sets). With general production sets (not just convex) as considered by Smale, his definition becomes essentially equivalent to the notion of "marginal cost pricing equilibrium" (Brown and Heal, 1979). This simply means that at an equilibrium the "equal gradient condition" is satisfied, although no optimization of profits is assured. Marginal cost pricing is a well-established principle for economies with nonconvex production, and it is widely used in practice for regulating industries with "increasing returns to scale," i.e., with nonconvex production. In addition, marginal cost pricing can be shown to be a necessary (but not sufficient) condition for efficiency of an equilibrium.

Obviously, marginal cost pricing equilibria need not satisfy Pareto optimality conditions in nonconvex economies. Such cases are included in Smale's existence work, as well as in his work on the local uniqueness of such equilibria [1974a], although he does not discuss the welfare properties of the nonconvex cases.

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[4] When an additional condition ensuring positive incomes is added. Such a condition is not necessary in convex economies.

Since Smale's existence work on economies with nonconvex production, we have learned that this failure of optimality can be quite serious: there exist nonconvex economies with several (locally unique) equilibria as defined by Smale, where all the equilibria fail to be Pareto optimal. This was proved by Guesnerie [1975] and by Brown and Heal [1979], following parallel lines which did not make contact with Smale's work. More recently, we have learned that this problem of failure of optimality is generic. Generically in nonconvex economies, all marginal cost pricing equilibria fail to be Pareto optimal (Chichilnisky, 1990).

Smale's work on nonconvex economies is not concerned with Pareto optimality. It appears to emerge from a natural mathematical position with respect to the limitations of convexity. While powerful, convexity is a rather special case, and it cannot be justified from the observation of economic reality. Smale's work, which focuses on a differentiable approach, naturally included convex as well as nonconvex cases. This simpler and more general approach to general equilibrium—which, as we noted, leads to unexpected failures of the optimality of equilibrium—is possibly one of the most interesting derivatives of Smale's work in general equilibrium theory. It seems fair to say that his work in this area has hardly been utilized in the economic literature, so that we may expect many more ramifications in the future.

Smale followed the path started by Von Neumann of utilizing topological methods to resolve simultaneous, conflicting, optimization problems arising from economics. Whereas Von Neumann introduced new models in economics, such as the Theory of Games and General Equilibrium Theory, Smale deepened and refined our understanding of existing models and open problems. He also provided a fruitful introduction of differential topology methods and showed the value of this machinery both for theoretical and applied purposes.

3. A Second Approach to Conflict Resolution in Economics: The Theory of Games

Smale also contributed to the theory of games, particularly in one paper studying the solutions of the "prisioner dilemma" game [1980]. Game theory provides a different approach to the resource-allocation problem. It typically assumes that the players have certain "strategies" available to them. The game is defined by specifying the outcomes which obtain when each player is playing a given strategy. This is called a "payoff function." A widely used solution concept is that of a Nash equilibrium (Nash, 1950). Here, each player chooses a strategy which maximizes his/her preferences for outcomes, *given the strategies played by the other players.* The fact that other player's strategies are taken as given is usually described by saying that the equilibrium is noncooperative. Instead, *cooperative solutions* involve communication between players, so that players influence each other strategies and, in this sense, coalitions are formed which decide the outcomes.

Game theoretical models differ from market models in that they have much less structure. This gives more flexibility to the theory, so that more situations can be represented, but obviously, fewer general results obtain. Because the problem is posed in a very wide framework, particular assumptions are made in each case to prove results, with the consequence that the results are far less general than those of the general equilibrium theory. In particular, Pareto efficiency is generally lost in noncooperative solutions. The trade-off is in the structure of the problem: market theory has far more structure and, therefore, far more results.

Yet, game theory is closely connected with the market-allocation problem. Indeed, one of the first existence theorems produced by G. Debreu for a general equilibrium proved the existence of a market equilibrium by using a proof due to Nash (1950) to establish the existence of an equilibrium of a game in which, in addition to the traders defined in Section 2, a "government" was one of the players (Debreu, 1952). In addition, a widely used concept of a cooperative game solution, the *core* of a market game, is closely associated with the market equilibrium. In an appropriately defined limiting sense, the two concepts coincide.

The problem tackled by Smale in game theory is the Pareto optimality of solutions to the prisioner's dilemma, the archetype example of a game where noncooperative solutions, the Nash equilibria, are Pareto inferior. A long-standing intuition about this game is that if the players played repeatedly, they would "learn" that the noncooperative solution is inferior and alter their behavior appropriately. The question is to formalize this learning process so that a new solution may be defined that leads to Pareto optimal outcomes.

Smale studies that questions by defining a repeated game, i.e., one which is played time and again, with finite memory. This is formalized by defining a time-dependent map from the set S of convex combination of all possible strategies to itself, $\beta_T: S \to S$, where $T \in N$ denotes time. The idea is to represent memory of the past by the average of past strategies. Given an initial strategy at time $T = 0$, at each T the map, β_T summarizes a time-dependent sequence of strategies, each generated by the average of the previous ones, thus defining dynamical systems associated to the noncooperative game. The interest of his approach is that he obtains Pareto efficient solutions asymptotically, the type of solutions which one may expect from a cooperative approach, by studying noncooperative solutions when the agents have memory. Again using topological tools, Smale uses his formulation to prove, under certain conditions, the existence of Pareto optimal solutions which are globally stable as usually defined in dynamical systems theory.

4. A Third Approach to Conflict Resolution: Social Choice Theory

Our last task is to explain the relation between Smale's work and social choice theory, an area in which Smale did not contribute directly.

Social choice theory starts by the assumption that there exists a set X of all possible choices or social states, which could be interpreted as the space of all possible allocations of the total social endowments among all m individuals. X is generally a closed, convex subset of R^l. Each individual i has a preference p_i defined on X, $i = 1, \ldots, m$, (see definition of $\leq$ in Section 2). The space of all individual preferences is denoted P; this is an important space with a non-trivial topological structure. As seen below, the solution of the social choice allocation problem hinges on its topology. The social choice problem is to find a "social preference" p, which ranks the elements in X, and, thus, can choose the social optimum among the feasible allocations in X.

It is desirable, and thus assumed, that the social preference depends on the individual preferences p_i, $i = 1, \ldots, m$, i.e., $p = \phi(p_1, \ldots, p_m)$. Equivalently, there must exist a function $\phi: P^m \to P$. The properties of the function ϕ determine the manner in which the social preference depends on the individual preferences. Ethical principles dictate that ϕ should be symmetric on the individuals, an "equal treatment" property, (i.e., *respect anonymity*), and that ϕ should be the identity map on the diagonal of P^m (meaning that ϕ *respects unanimity*). Practicality requires that ϕ be *continuous*; this assures the existence of sufficient statistics, namely, "polls" taken from finite samples which yield information approximating the true function. These three axioms: continuity, anonymity, and respect of unanimity were introduced in Chichilnisky [1980]; they relate to, but are different from, other axioms used in the Social Choice literature commenced by Kenneth Arrow's work.

In general, the three axioms, continuity, anonymity and respect of unanimity, are inconsistent with each other (Chichilnisky, 1980) and certain restrictions are necessary for social choice allocations satisfying these axioms to be possible (Chichilnisky and Heal, 1983). This means that a function $\phi: P^m \to P$ respecting all three axioms does not exist when P is the space of all preferences Chichilnisky (1980). The nonexistence result hinges on the topological structure of the space of preferences P. This became clear when, after a substantial literature developed on this problem, Chichilnisky and Heal (1983) proved that for any given (CW complex) space X a map $\phi: X^m \to X$ exists for all m if and only if the homotopy groups $\Pi_j(X) = 0$ for all $j \geq 1$ (Theorem 3 below). This is equivalent to X being contractible, i.e., topologically equivalent to a point: therefore, the allocation problem in economics, posed in terms of a social choice, is essentially a topological issue, one which can be resolved if and only if there is no topological complexity.

This section will show that the existence of the competitive equilibrium has the same characteristics as the social choice allocation problem: It will be shown that the topological obstruction for the social choice resolution and for the existence of an equilibrium are one and the same: They lie in the topology of a family of cones which are naturally associated to the economy's preferences..

We need some notation, in particular we must define the space of preferences and explain how it differs from a vector space of real-valued functions. The latter, being linear, is, of course, topologically trivial.

The following is a definition of preferences equivalent to that of Section 2.

Consider the space of real-valued functions on $X, f: X \to R$, with the equivalence relation $f \approx g$ if and only if $\forall x, y \in X, f(y) \geq f(x) \leftrightarrow g(y) \geq g(x)$. The space P of all preferences is then the quotient space of the space F of all real-valued functions on X under the equivalence relation $\approx$, $F/\approx$. With appropriate smoothness and regularity restrictions on F ($f \in F \to f$ is C^2 and has no critical points) such as those used is Smale's work, one considers P as the space of all C^1 unit vector fields $v: X \to R^l$ which are globally integrable, i.e., essentially the space of codimension—one C^1 oriented foliations of X without singularities (Chichilnisky, 1980).

Recent results have been used to establish the topological connection between the problem of existence of a general equilibrium to which Smale contributed extensively, and the social choice solution of the resource allocation conflict in economics in Chichilnisky [1991]. The rest of this section will explain this connection.

4.2. Social Choice and the Existence of a Competitive Market Equilibrium

Consider the exchange general equilibrium model defined in Section 2. Typically, it is assumed that consumption sets X_i are positive orthants. Here, instead, we assume that the individual consumption sets X_i are all of R^l; this is elaborated further below. In this case, it is possible to give necessary and sufficient condition a simple for the existence of a general equilibrium (Chichilnisky, 1992), defined on the properties of a family of cones which represent the relationships between the different individual preferences. These necessary and sufficient conditions are in turn also necessary and sufficient for the resolution of the social choice problem as defined by Chichilnisky [1980] and Chichilnisky and Heal [1983].

Consider the pure exchange general equilibrium model as defined in Section 2 with one further specification: for each consumer i, the consumption set X_i, which is the set of all consumption vectors which are feasible to the consumer, is the whole commodity space R^l. Such consumption sets are familiar in financial markets: they include "short" sales of commodities, which are sales involving a quantity larger than what is owned by the seller. When there are no limitations on the coordinates of the consumption vectors,[5] x_i, $\forall i, X_i = R^l$.

[5] Debreu [1993] mentions that for each consumer, the consumer inputs are positive numbers, and outputs are negative numbers. The hth coordinate of a vector x_i in the consumption set X_i is then the quantity of the hth commodity that he/she consumes if $x_i^h > 0$, or the negative of the quantity he/she produces if $x_i^h < 0$. For example, if a consumer's output is the amount of labor he/she contributes, because labor output is limited to 24 hours a day, then the negative coordinates of the consumption set X_i are bounded below. Such restrictions are however not applicable to financial markets, where in principle X_i could be unbounded below, and, in fact, the whole of the space R^l, see also Chichilnisky and Heal [1984, 1993].

When each consumer's consumption set X_i is the whole commodity space R^l rather than R^{l+}, certain conditions on individual preferences are needed for the existence of a competitive equilibrium without which it fails to exist. These conditions replace the boundary conditions used by Smale to prove existence. Arrow–Debreu consider for example the model defined in Section 2, where commodities are also indexed by date and state of nature so that, for example, an apple tomorrow if it rains is different from an apple today, and also different from an apple tomorrow if it does not rain, and they all have potentially different prices. In such models, consumer preferences include their subjective probabilities for the states of the world in the next period. It is clear that with two traders who have the opposite expectations about the price of apples tomorrow (for example, one expecting that with probability one the price of apples will increase, and the other the opposite, also with probability one), they will always increase their utility by trading increasing amounts in the opposite direction for delivery tomorrow. The trader who expects prices to drop will want to sell short, any amount, and the trader who expects prices to increase will want to buy more, any amount. Thus, in this two-person economy, an equilibrium will not exist unless an exogenous bound is introduced on "short" apple sales, such as, for example, a lower bound on the consumption sets X_i of the traders.

Such lower bounds are obviously artificial. In addition, the equilibria found by imposing exogenously such lower bounds, depend, of course, on the chosen bound, leading to indeterminacy of the model (i.e., which bound to choose?).

For this reason, it is preferable to postulate conditions on preferences, which are the primitives of the model, to assure the existence of an equilibrium. This is what has been done in Chichilnisky and Heal [1984, 1993]. We pursue a similar approach here, except that here we shall provide conditions which are "tighter" being both necessary and sufficient for the existence of an equilibrium.

We need some definitions. Let $E = \{(X_i), (\leq_i), (e_i), i = 1, \ldots, m\}$ be an economy as in Section 2, with $\forall i$, $X_i = R^l$. Define the ith trader *preferred cone* $A_i = \{v \in R^l\colon \forall x \in R^l \exists \lambda > 0\colon (e_i + \lambda v) >_i (x_i)\}$, and its *dual* $D_i = \{v \in R^l\colon \forall y \in A_i, \langle y, v\rangle > 0\}$. We shall say that a smooth preference p is consistent with the cone A_i if $\forall x_i \in R^l$ and $\forall y \in A_i$, $\langle p(x_i), y\rangle > 0$, where $p(x)$ is the vector field defining the preference p. This means that the preference p increases in the direction of the preferred cone A_i. For any subset of traders $\theta \subset \{1, \ldots, m\}$, a smooth preference p is similar with those of the *subeconomy* $E_\theta = \{(X_i), (\leq_i), (e_i), i \in \theta\}$ if it is consistent with the preferred cone A_i for some $i \in \theta$. The space of all preferences similar to those of the subeconomy E_θ is denoted P_θ. Define now the following conditions:

(A) The intersection of the duals $\bigcap_{i=1}^m D_i$ is not empty.
(B) The union $\bigcup_{i \in \theta} D_i$ is contractible for all $\theta \subset \{1, \ldots, m\}$.

Then we have the following results:

Theorem 1. *Conditions* (A) *and* (B) *are equivalent* (*Chichilnisky*, 1981, 1991).

Theorem 2. *Consider the pure exchange economy* $E = \{(X_i), (\leq_i), (e_i), i = 1, \ldots, m\}$ *of Section 2. Assume that all standard conditions on preferences are satisfied:* $\forall i$, $\leq_i$ *is smooth, concave and increasing. The consumption sets are* $X_i = R^l$ *for all i. Then condition* (A) *is necessary and sufficient for the existence of a competitive equilibrium* (*Chichilnisky*, 1992).

If X is a CW complex:

Theorem 3. *A continuous anonymous map* $\Phi: X^k \to X$ *respecting unanimity exists for all* $k \geq 1$ *if and only if* $\pi_j(X) = 0$ *for all* j. *In particular,* X *is contractible.*

From Theorems 1, 3 and Chichilnisky (1991):

Theorem 4. *For each subset of traders* $\theta \subset \{1, \ldots, m\}$, *consider the space* P_θ *of all smooth preferences on* R^l *which are similar to those in the subeconomy* E_θ. *Then there exists a continuous anonymous map* $\Phi: P_\theta^k \to P_\theta$ *which respects unanimity for all* $k \geq 1$ *and* θ, *if and only if* (B) *is satisfied.*

Theorems 1, 2 and 4 imply:

Theorem 5. *The economy* $E = \{(X_i), (\leq_i), (e_i), i = 1, \ldots, m\}$ *of Theorem 2 has a competitive equilibrium if and only if for each subset of traders* $\theta \subset \{1, \ldots, m\}$ *the space of preferences* P_θ *similar to those of the subeconomy* E_θ *admits a continuous anonymous map* $\Phi: P_\theta^k \to P_\theta$ *respecting unanimity,* $\forall k \geq 1$.

This last theorem proves an equivalence between two topological problems, the existence of a general equilibrium and the resolution of the social choice problem, and completes our arguments.

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