

## INTERSECTING FAMILIES OF SETS AND THE TOPOLOGY OF CONES IN ECONOMICS

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**ABSTRACT.** Two classical problems in economics, the existence of a market equilibrium and the existence of social choice functions, are formalized here by the properties of a family of cones associated with the economy. It was recently established that a necessary and sufficient condition for solving the former is the nonempty intersection of the family of cones, and one such condition for solving the latter is the acyclicity of the unions of its subfamilies. We show an unexpected but clear connection between the two problems by establishing a duality property of the homology groups of the nerve defined by the family of cones. In particular, we prove that the intersection of the family of cones is nonempty if and only if every subfamily has acyclic unions, thus identifying the two conditions that solve the two economic problems. In addition to their applications to economics, the results are shown to extend significantly several classical theorems, providing unified and simple proofs: Helly's theorem, Caratheodory's representation theorem, the Knaster-Kuratowski-Marzukiewicz theorem, Brouwer's fixed point theorem, and Leray's theorem on acyclic covers.

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### 1. INTRODUCTION

A classic problem in economics is the existence of a *market equilibrium* (Von Neumann [37], Nash [32]). This can be viewed as a zero of a nonlinear map  $\Psi : R^N \rightarrow R^N$  representing market excess demand and embodying optimal behavior of the traders (Arrow and Debreu [3]). The zero can be located by homotopy methods (Eaves [23], Hirsch and Smale [30]). Smale [34, 35] has

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reexamined an intuitively appealing dynamical system which is compatible with a field of cones of directions of improvement for the economy. Along its solution paths all traders gain and proceed until no more gains can be attained and an equilibrium is reached. However, unless the economy satisfies strong boundary conditions, this process may not converge and the market equilibrium may fail to exist.

Another classic problem in economics is the existence of *social choice functions*, (Arrow [5]). These can be viewed (Chichilnisky [8]) as maps which assign to each vector of individual preferences a social preference,  $\Phi : P^k \rightarrow P$ , where  $P$  is the space of preferences and  $k$  is the number of individuals.  $\Phi$  must satisfy certain properties which derive from ethical considerations such as symmetry, an equal treatment condition. The problem has a clear topological structure. A map  $\Phi$  exists for a given  $k$  only when a certain topological obstruction disappears. It exists for all  $k$  if and only if the space  $P$  is topologically trivial (Chichilnisky and Heal [15]). In general, the space  $P$  is infinite dimensional and has nontrivial homology, so a social choice rule may fail to exist [8, 14].

Both problems are fundamental to the organization of society. Their solutions model social agreements about how to allocate the resources of the economy among competing individuals, the market solution providing an allocation which is *efficient* (Arrow [2]) and the social choice solution one which satisfies certain ethical properties. The solutions represent different types of "social contracts".

While these two problems appear to be quite different and have been considered separately until now, we show that, in a well-defined sense, they are the same. We provide here a topological formulation of these problems which allows us to identify each with apparently different properties of a family of cones which is naturally associated with the economy. It was recently shown that the existence of a competitive equilibrium requires the family of cones to intersect; the existence of social choice functions requires that all subfamilies have acyclic unions (Chichilnisky [12, 13]). Looking at the problem in its simplest and most general form, we obtain a topological characterization of a family of finitely many sets in a general topological space that is necessary and sufficient for the family to have a nonempty intersection<sup>1</sup>. One main result is that an acyclic or convex family has nonempty intersection if and only if every subfamily has acyclic union (Theorem 6 and Corollary 2), but the results extend to nonacyclic, nonconvex families as well (Theorems 9 and 10). As a by-product, we establish the identity between the two classical problems in economics, namely, the existence of a social choice function and of a competitive equilibrium (Theorem 11).

The topology of our family of cones contains crucial information about the economy. The homology of its nerve defines a topological invariant for the economy which provides answers to global problems such as, for example, whether a market equilibrium exists (Theorems 1, Corollary 2, and Theorem 11). Furthermore, this invariant allows us to decide whether every subeconomy has a

<sup>1</sup>This result was first established in Chichilnisky [9].

competitive equilibrium (Theorem 11(b)). The homology of this nerve also contains information about the global convergence of the classic price adjustment process in Smale [34, 35] (see Chichilnisky [18])—it determines whether this process converges.

The homology of the nerve of a family of sets also provides valuable information in a number of other applications in fields other than economics, which appear as additional by-products of the results in this paper. These include substantial extensions and unified proofs for classical theorems which have until now been considered disparate: Helly's theorem on  $n + k$  convex sets in  $R^n$ ,  $k > 1$  ([27, 28, 1]), which is used extensively in game theory, for example, Guesnerie and Oudu [25]; Caratheodory's theorem and its relative the Krein-Milman theorem, both of which are used in representation theory to characterize the extreme elements of the cone of positive harmonic functions on the interior of the disk (Choquet [20]); the Knaster-Kuratowski-Marzukiewicz (KKM) theorem (Berge [6]), which is frequently used to prove the existence of the core of a game (Scarf [33]); the Brouwer fixed point theorem, which is the nonretractability of a cell onto its boundary and is used to prove existence of solutions of simultaneous equations (Hirsch [29], Arrow and Hahn [4]); and Leray's theorem on the isomorphism between the homology groups of a space and those of the nerves of an acyclic cover (Leray [31], Dowker [22], Cartan [7]). These classical theorems of Helly, Caratheodory, Leray, and KKM are extended here to simple and regular families of arbitrary finite cardinality, consisting of sets which need not be open nor acyclic or even connected and which are contained in general topological spaces, including infinite-dimensional spaces; our results generalize also the Brouwer's fixed point theorem which appears as an immediate corollary. In addition, our topological approach allows us to obtain conditions which are simultaneously necessary and sufficient for nonempty intersection of a general family of sets (Chichilnisky [9]), a result which we find here very useful and which was not available before.

Here is a summary of the paper. In §§2–4 we set out the context and describe the problems of existence of a market equilibrium and of a social choice function. A necessary and sufficient condition for the existence of a market equilibrium—called *limited arbitrage*—is defined as the nonempty intersection of a family of cones. A necessary and sufficient condition for the existence of social choice functions—called *limited social diversity*—is defined as the acyclicity of the unions of subfamilies of the same family of cones. Our task is to prove that the two conditions are in fact identical. This identity (Theorem 11) is a corollary of the results in §5.

Section 5 studies the problem in a general form. First we prove a duality result which relates the reduced singular homology groups of the union and the intersection of a subfamily in dimensions which are complementary with respect to its cardinality [9]. This analysis is used to prove that all subfamilies up to a certain cardinality have acyclic unions if and only if they have acyclic intersections. Then we establish that the whole family has a nonempty acyclic intersection if and only if all the reduced homology groups of the union of its subfamilies up to a certain cardinality vanish.

We further extend the results to families of sets which need not be open, acyclic, or even connected in order to obtain a condition for the nonempty

intersection of the family, whether or not this intersection is acyclic. The results thus provide a topological characterization of families of sets which have a nonempty intersection. In particular, this characterization shows that a convex family has a nonempty intersection if and only if all its subfamilies have acyclic unions. Therefore, *limited arbitrage* is identical to *limited diversity*, and the problems of existence of a competitive equilibrium and of social choice functions are the same.

Sections 6 and 7 apply the results in §5 to extend a number of classical theorems and to provide simple, unified proofs to such disparate results as Helly's theorem, Caratheodory's representation theorem, the Knaster-Kuratowski-Marzukiewicz theorem, Brouwer's fixed point theorem, and Leray's theorem on acyclic covers. Our extensions of these classical results include families of sets in arbitrary topological spaces to which the earlier results do not apply, sets which need not be open, convex, acyclic, or even connected. The families may, in addition, be of arbitrary finite cardinality. Section 7 establishes the identity between the problem of existence of a competitive equilibrium and the problem of existence of social choice functions.

## 2. DEFINITIONS

We consider collections of finitely many sets in a topological space  $X$ , denoted  $\{U_\alpha\}_{\alpha \in S}$ , with set of indices  $S$ . Such a collection is called a *cover* of  $X$  when  $X = \bigcup_{\alpha \in S} U_\alpha$ ; it is an *open cover* when each set is open in  $X$ . The term *family* will be used to describe a collection of finitely many sets  $\{U_\alpha\}_{\alpha \in S}$  in  $X$  whose union  $\bigcup_{\alpha \in S} U_\alpha$  may or may not cover  $X$ . An *open family* in  $X$  is a family consisting of sets which are open in  $X$ . A subset of indices in  $S$  will be indicated by  $\theta \subset S$ ; each subset  $\theta \subset S$  defines a *subfamily*  $\{U_\alpha\}_{\alpha \in \theta}$  of the *family*  $\{U_\alpha\}_{\alpha \in S}$ . We shall use the notation  $U_\theta$  for the intersection of the subfamily indexed by  $\theta$ ,  $U_\theta = \bigcap_{\alpha \in \theta} U_\alpha$ , and  $U^\theta$  for its union  $U^\theta = \bigcup_{\alpha \in \theta} U_\alpha$ .

$H_*$  will be used to denote *reduced* singular homology, and  $H_q(Y)$  to denote the  $q$ -singular reduced homology group of the space  $Y$ ; *reduced* singular homology is defined by replacing the usual chain complex

$$\cdots C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

by

$$\cdots C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow Z \rightarrow 0,$$

where  $Z$  are the integers and  $C_0 \rightarrow Z$  takes each 0-simplex to 1. The corresponding reduced singular homology groups denoted  $H_q(Y)$  are defined for all  $q \geq -1$ . The standard 0-singular homology of  $Y$  is the direct sum  $H_0(Y) \oplus Z$ . Note that with this notation if  $Y$  is a nonempty connected space, then  $H_0(Y) = 0$  and  $H_{-1}(Y) = 0$ ; and if  $Y$  has two connected components, then  $H_0(Y) = Z$ . If  $Y$  is empty,  $H_0(Y) = 0$  and  $H_{-1}(Y) = Z$ . It is immediate that with this definition the Mayer-Vietoris sequence (Spanier [36, §6, Chapter 4]) extended to *reduced* singular homology

$$\cdots H_{q+1}(A \cap B) \rightarrow H_{q+1}(A) \oplus H_{q+1}(B) \rightarrow H_{q+1}(A \cup B) \rightarrow H_q(A \cap B) \rightarrow \cdots$$

is exact.

We say that a space  $Y$  is *acyclic* if and only if  $H_*(Y) = 0$ . Since by definition the space  $Y$  is nonempty if and only if  $H_{-1}(Y) = 0$ , in our notation  $Y$  is called

acyclic when  $Y$  is not empty and is acyclic in the standard singular homology. When the space  $X$  is contained in a linear space, a family is called *convex* if it consists of convex sets. A family  $\{U_\alpha\}_{\alpha \in S}$  is called *acyclic* if, for all  $\theta \subset S$ , the set  $U_\theta$  is either empty or acyclic.

For any  $k \geq 0$  we say that the family  $\{U_\alpha\}_{\alpha \in S}$  satisfies **condition  $A_k$**  if the intersection  $U_\theta$  is acyclic for every  $\theta \subset S$  having at most  $k + 1$  elements.

For any  $k \geq 0$  we say that the family  $\{U_\alpha\}_{\alpha \in S}$  satisfies **condition  $B_k$**  if the union  $U^\theta$  is acyclic for every  $\theta$  having at most  $k + 1$  elements.

If  $X \subset R^n$ , then the family  $\{U_\alpha\}_{\alpha \in S}$  is called a *family in  $R^n$*  and is called a *family of  $k$  sets* if  $S$  has cardinality  $k$ .

If  $X$  is a simplicial complex with set of vertices  $S$ , then a *simple cover* of  $X$  is an *open cover*  $\{U_\alpha\}_{\alpha \in S}$  of  $X$  satisfying  $\text{cl}(U_\alpha) \subset \text{star}(\alpha)$  for all  $\alpha \in S$ , where  $\text{cl}(Y)$  is the closure of  $Y$  and  $\text{star}(\alpha)$  is the interior in  $X$  of the union of all closed simplices in  $X$  having  $\alpha$  as a vertex.

The sets in a simple family need not be convex nor acyclic or even connected. A subcomplex  $L$  of a simplicial complex  $K$  is a subset of  $K$  (that is, if  $s \in L \Rightarrow s \in K$ ); a subcomplex  $L$  is called *full* if each simplex of  $K$  having all its vertices in  $L$  belongs to  $L$  (Spanier [36]). The symbol  $[\alpha]_{\alpha \in \theta}$  denotes the *full subcomplex* of  $X$  with set of vertices  $\{\alpha\}_{\alpha \in \theta}$ .

A cover of the simplicial complex  $X$  by finitely many *closed* sets  $\{C_\alpha\}_{\alpha \in S}$  is called *regular* if  $\forall \theta \subset S$ ,  $[\alpha]_{\alpha \in \theta} \subset \bigcup_{\alpha \in \theta} C_\alpha$ .

A regular cover  $\{C_\alpha\}_{\alpha \in S}$  of a simplicial complex  $X$  therefore satisfies: for every subset  $\theta \subset S$  and every simplex  $\Delta$  of  $X$  whose vertices lie in  $\bigcup_{\alpha \in \theta} C_\alpha$ , we have  $\Delta \subset \bigcup_{\alpha \in \theta} C_\alpha$ . The sets in a regular cover need not be convex, acyclic, or even connected.

Given a set  $X$  and a collection  $\{U_\alpha\}_{\alpha \in S}$  of subsets of  $X$ , the *nerve* of  $\{U_\alpha\}_{\alpha \in S}$  is the simplicial complex having as vertices the nonempty elements of  $\{U_\alpha\}_{\alpha \in S}$  and whose simplexes are finite nonempty subsets of  $\{U_\alpha\}_{\alpha \in S}$  with nonempty intersection (Spanier [36]).

### 3. MARKET EQUILIBRIUM

**3.1. A market economy.** A market economy is described by its *goods* and its *traders*. There are  $n > 1$  goods and  $H > 1$  traders. Traders derive utility from vectors (called *trades* or *bundles of goods*) in  $R^n$ , which is called the *consumption* or *trade space*. Each trader is identified by a vector describing his/her initial *endowments* of goods  $\Omega_h \in R^n - \{0\}$  and by a real-valued smooth ( $C^2$ ) function  $u_i : R^n \rightarrow R$  which describes the *utility* derived from the different consumption vectors. The *space of allocations* is  $R^{nH}$ ; its elements describe the assignment of one consumption vector in  $R^n$  for each trader. The utilities  $u_i$  are increasing:  $\forall x, y \in R^n$ , if  $x \geq y$ , then  $u_i(x) \geq u_i(y)$ , and  $\exists \epsilon > 0 : Du_i(x) > \epsilon$ , where  $Du_i(x)$  is the gradient vector of  $u_i$  at  $x$ . If for some  $r \in R$  the set  $u_i^{-1}(r, \infty)$  is not bounded below in  $R^n$ , then we assume that the set of directions of gradients of the corresponding hypersurface,  $\{v = Du_i(x) / \|Du_i(x)\| : u_i(x) = r\}$ , is closed in  $R^n$ . This assumption is to

control the behavior at infinity of the leaves of the foliation of  $R^n$  induced by the hypersurfaces of the function  $u_i$ ; geometrically, one rules out "asymptotic directions" for the gradients on those hypersurfaces which are not bounded below. A *market economy*  $E$  is therefore defined by its trade space and its traders:  $E = \{R^n, \Omega_i, u_i, i = 1, \dots, H\}$ .

**3.2. Market equilibrium.** Our next tasks are to motivate and then to define the notion of a *competitive equilibrium for the market*  $E$ . A competitive equilibrium represents a rest point of the trading activity of the economy  $E$ . Trading requires *prices*. A price is a rule which assigns a real number called *value* to each bundle of goods in a way that depends linearly on the bundles. Therefore, prices are vectors in the dual space of the space of trades,  $R^n$ . Each price  $p \in R^n$  determines the *budget set* of a trader  $B(p, \Omega_i)$  consisting of those trades which are affordable at the traders' initial endowment  $\Omega_i$ . Therefore,  $B(p, \Omega_i) = \{x \in R^n : \langle p, x \rangle = \langle p, \Omega_i \rangle\}$ , where  $\langle \cdot, \cdot \rangle$  is the inner product in  $R^n$ . Traders trade within their budgets in order to increase, ideally to optimize, their utility.

Trading comes to a rest when a price  $p^* \in R^n$  is found at which the corresponding set of all optimal trades  $\{x_i^*\}_{i=1, \dots, H}$  is compatible with the resources of the economy, i.e., the *supply* of each of the  $n$  goods equals the *demand*. A *competitive equilibrium* of the market economy  $E$  is therefore defined as a vector of prices and of trades,  $(p^*, x_1^* \dots x_H^*) \in R^n \times R^{nH}$ , satisfying the following conditions:

$$(1) \quad u_i(x_i^*) = \text{Max}_{x_i \in B(p^*, \Omega_i)} u_i(x_i)$$

$$\text{for } B(p^*, \Omega_i) = \{x \in R^n : \langle p^*, x \rangle = \langle p^*, \Omega_i \rangle\}$$

and

$$(2) \quad \sum_{i=1}^H (x_i^* - \Omega_i) = 0 \in R^n.$$

The vector  $x_i^*(p^*)$  is the *demand* of trader  $i$  at prices  $p^*$ ; a solution  $x_i(p)$  to problem (1) for all  $p \in R^n$  is the *demand function*  $x_i(p) : R^n \rightarrow R^n$  of trader  $i$ .  $ED(p) = \sum_{i=1}^H (x_i(p) - \Omega_i)$  is the *aggregate excess demand function*<sup>2</sup> of the economy  $E$ . Condition (2) means that at the equilibrium allocation all markets clear, i.e., total demand for each good equals total supply, and therefore  $ED(p^*) = 0$ .

**3.3. Market cones.** Consider a market economy  $E = \{R^n, \Omega_i, u_i, i = 1, \dots, H\}$ . The *asymptotic preferred cone*  $A_i$  is the cone of all directions which intersect every hypersurface of  $u_i$  of values exceeding  $u_i(\Omega_i)$ :

$$(3) \quad A_i = \{v \in R^n : \sup_{\lambda \in (0, \infty)} u_i(\Omega_i + \lambda v) = \sup_{x \in R^n} u_i(x)\}.$$

The *market cone*  $D_i$  is

$$(4) \quad D_i = \{p \in R^n : \forall v \in A_i, \langle p, v \rangle > 0\}.$$

<sup>2</sup>The demand and the aggregate excess demand functions may not be well defined for some prices which are not equilibrium prices.

If the utility  $u_i$  is a concave function, then both cones  $A_i$  and  $D_i$  are open convex sets, which we now assume. The condition of *limited arbitrage* (LA) is that all *market cones* in (4) intersect:

$$(5) \quad (\text{LA}) \quad \bigcap_{i=1}^H D_i \neq \emptyset.$$

This means that there exists a price  $p \in R^n$  at which only limited increases in utility can be achieved by all traders from trades which are affordable from their initial endowments.

The following has been established:

**Theorem 1.** *Limited arbitrage (5) is necessary and sufficient for the existence of a competitive equilibrium in the market  $E$ .*

For a proof see Chichilnisky [12].

The condition for existence of a competitive equilibrium is therefore the nonempty intersection (5) of a family of cones in  $R^n$  which are naturally associated with the economy  $E$ , namely, of the family of market cones  $\{D_i\}_{i=1, \dots, H}$  defined in (4). The market cones  $\{D_i\}_{i=1, \dots, H}$  contain global information about the economy, since they establish directions of utility increases along which all utility levels are eventually reached. As established in Theorem 1, the market cones  $\{D_i\}_{i=1, \dots, H}$  determine whether or not the market has a competitive equilibrium. They also determine whether or not the dynamical process revisited in [34, 35] converges globally; it converges if and only if limited arbitrage holds, i.e., if and only if the family of cones has nonempty intersection (see Chichilnisky [18]).

The family of market cones  $\{D_i\}_{i=1, \dots, H}$  also contains information about the existence of social choice functions. In the next section we shall see that a condition for existence of a social choice function is that every subfamily of the family of market cones,  $\{D_i\}_{i=1, \dots, H}$ , has an acyclic union.

#### 4. SOCIAL CHOICE FUNCTIONS

**4.1. Individual and social preferences.** In this section we consider a connected and simply connected  $CW$  complex  $P$  (Spanier [36]) representing a space of preferences on  $R^n$ . The explicit cell structure on  $P$  is not needed, only the general topological properties of  $CW$  complexes. For example,  $P$  could be a polyhedron or a smooth manifold.  $P^k$  denotes the product of  $P$  with itself

$k$  times,  $P^k = \overbrace{P \times \dots \times P}^{k \text{ times}}$ , and  $\Delta P$  is the "diagonal" of  $P^k = \{(p_1 \dots p_k) \in P^k : \forall i, j = 1, \dots, k, p_i = p_j\}$ . Examples of spaces of preferences  $P$  are provided in §7.

**4.2. Social choice functions.** A *social choice function* for the space of preferences  $P$  and for  $k$  individuals, is a continuous map  $\Phi : P^k \rightarrow P$  assigning to each vector of  $k$  individual preferences in  $P^k$  a *social preference* in  $P$  satisfying:

1.  $\Phi$  is symmetric; i.e.,  $\Phi$  is invariant under the action of the group of permutations of  $k$  letters acting naturally on  $P^k$ .

This condition means that all  $k$  individuals are treated equally and is called *anonymity*.

2. The map induced by the restriction of  $\Phi$  on  $\Delta(P^k)$  at the homotopy level,  $(\Phi|_{\Delta(P^k)})_* : \pi_j(\Delta(P^k)) \rightarrow \pi_j(P)$ , is onto  $\forall j$ .

This condition arises from several applications [14, 16]. For example, it is implied by the *Pareto condition* [10], which requires that when all individuals prefer one choice  $x$  to another  $y$ , so does society. It is also implied by the assumption that  $\Phi|_{\Delta(P^k)} = \text{id}(\Delta(P^k))$ ; i.e., when  $\Phi$  is restricted to the "diagonal" of  $P^k$ ,  $\Delta P^k = \{(\rho_1 \cdots \rho_k) \in P^k \text{ s.t. } \forall i, j, \rho_i = \rho_j\}$ , it is the identity map. This latter condition means that when all individuals have the same preference, society adopts that common preference, and it is called *respect of unanimity* [8].

An allocation is an assignment of a bundle of goods in  $R^n$  to each trader, and the space of allocations is  $R^{nH}$ . Each trader has a preference over allocations. A smooth preference over the space of allocations  $R^{nH}$  is a smooth ( $C^2$ ) unit vector field  $\rho : R^{nH} \rightarrow S^{nH-1}$  satisfying:  $\exists u : R^{nH} \rightarrow R$  with  $\forall x \in R^{nH}$ ,  $\rho(x) = \lambda(x) Du(x)$  for some  $\lambda(x) > 0$  (Debreu [21]). The space of all smooth preferences on allocations in  $R^{nH}$  is denoted  $\Gamma(R^{nH})$ . The space of preferences  $P(E_\theta)$  similar to those of a subset  $\theta \subset \{1, \dots, H\}$  of traders in  $E$  is

$$(6) \quad P(E_\theta) = \left\{ \rho \in \Gamma(R^{nH}) : \forall x \in R^{nH} \text{ and } \forall j, \rho^j(x) \in \bigcup_{i \in \theta} D_i \right\},$$

where  $\rho^j(x)$  is the projection of  $\rho(x)$  on the  $j$ th copy of  $R^n$  in the product space  $R^{nH}$ . The interpretation is that  $P(E_\theta)$  consists of all preferences which are similar to those of some trader  $i \in \theta$  in some position  $j$  in the sense that they increase in the directions of large utility increases for  $i$  in position  $j$  and only in those directions. This is discussed further in §7. Note that the notion of similarity of preferences depends on the same family of market cones  $\{D_i\}_{i=1, \dots, H}$  defined in equation (4) in §3.

**4.3. Social choice and the topology of preferences.** In its most general form the problem of existence of social choice functions has no solution; for the space  $\Gamma = \Gamma(R^m)$  of all smooth preferences on  $R^m$ ,  $m > 2$ :

**Theorem 2.** *There exists no map  $\Phi : \Gamma^k \rightarrow \Gamma$  satisfying 4.2.1 and 4.2.2  $\forall k \geq 1$ .*

A proof is in Chichilnisky [8, 10].

A natural question is what spaces of preferences  $P$  admit a social choice function. The following is known:

**Theorem 3.** *There exists a social choice map  $\Phi : P^k \rightarrow P$  satisfying 4.2.1 and 4.2.2  $\forall k \geq 1$ , if and only if  $P$  is acyclic.*

This was proved in Chichilnisky [8] and Chichilnisky and Heal [15].

When a social choice function  $\Phi : P^k \rightarrow P$  exists, then by Whitehead's theorem (Spanier [36])  $P$  is contractible, since the space  $P$  is acyclic and by assumption  $\pi_1(P) = 0$ . Therefore, there exists a continuous deformation of the space of preferences  $P$  into one preference. For this reason, in this context the acyclicity of a space of preferences establishes a *limit on social diversity* (Heal [26]). For any given subset  $\theta$  of traders in  $E$ ,  $\theta \subset \{1, \dots, H\}$ , a



social choice function  $\Phi : (P(E_\theta))^k \rightarrow P(E_\theta)$  exists satisfying the required conditions  $\forall k > 1$  if and only if  $P(E_\theta)$  is acyclic. This in turn means that the space of gradients of the preferences in  $P(E_\theta)$ , namely,  $\bigcup_{i \in \theta} D_i$ , must be acyclic. We say the market  $E$  has *limited social diversity* or simply *limited diversity* (LS), when:

$$(7) \quad (\text{LS}) \quad \forall \theta \subset \{1, \dots, H\}, \quad \theta \neq \emptyset \Rightarrow \bigcup_{i \in \theta} D_i \text{ is acyclic.}$$

A consequence of Theorem 3 is:

**Theorem 4.** *There exists a social choice function  $\Phi : (P(E_\theta))^k \rightarrow P(E_\theta)$  satisfying 4.2.1 and 4.2.2,  $\forall \theta \subset \{1, \dots, H\}$  and  $\forall k \geq 1$ , if and only if the market  $E$  has limited social diversity (LS).*

This follows from Chichilnisky [8] and Chichilnisky and Heal [15].

**4.4. Social choice and the nerve of market cones.** For a social choice function  $\Phi$  to exist, the union of every nonempty subfamily of market cones  $\{U_i\}_{i=1, \dots, H}$  must be acyclic. We saw in §3 that the existence of a competitive equilibrium requires the nonempty intersection of the same family of market cones,  $\bigcap_{i=1}^H \{D_i\} \neq \emptyset$ . To identify the two economic problems, we must exhibit the connection between two properties of the family of cones. One is that the family has nonempty intersection—i.e., *limited arbitrage* (5). The second is that the union of every subfamily is acyclic—i.e., *limited diversity* (7). This is achieved in Theorem 11 in §7 and motivates the results in the following section.

## 5. DUALITY AND INTERSECTING FAMILIES

Having established the importance in economics of the topology of the nerve of the *market cones*  $\{D_i\}_{i=1, \dots, H}$ , we turn now to the mathematical problem. In their simplest and most general form the questions are: when does the family of market cones  $\{D_i\}_{i=1, \dots, H}$  have a nonempty intersection, and how does this relate to the acyclicity of the unions of its subfamilies? The nonempty intersection of this family of cones is the condition of *limited arbitrage* (5), and the acyclicity of the unions of its (nonempty) subfamilies is the condition of *limited diversity* (7). We saw in §3 that the former (5) is necessary and sufficient for the existence of a market equilibrium and in §4 that the latter (7) is necessary and sufficient for the existence of social choice functions. This section will establish inter alia that the two mathematical conditions (5) and (7) are identical.

Here is a summary of the section.<sup>3</sup> Theorem 5 proves the equivalence between two topological conditions of the nerve of a family of sets of a general topological space  $X$ —these are conditions  $A_k$  and  $B_k$  defined in §2, the former requiring that all subfamilies with at most  $k + 1$  elements have acyclic intersection and the latter requiring that all such subfamilies have acyclic unions. This identity is simple and geometrically appealing. It has many implications, as we show below. Because it is close to the foundations of homology theory, there is a subtle point in its proof, which ensures an excision property for singular

<sup>3</sup>The results in this section were first established in Chichilnisky (1981).

reduced homology (see, e.g., Spanier [36, p. 189]) so that the Mayer-Vietoris sequence for reduced singular homology—a sequence which is rarely used for families where the sets may have empty intersection—is exact. A discussion of this exactness for reduced homology for families which includes empty sets is in §2, and the excision property is discussed in this section after condition (6).

The exactness of the Mayer-Vietoris sequence is used in our proof of a duality property of the singular reduced homology of a family of sets in Proposition 1. This proposition establishes a simple isomorphism between the singular reduced homology groups of the union and those of the intersection of a subfamily in dimensions complementary with its cardinality. This duality property allows us to prove the following somewhat surprising result in Proposition 2: For families in  $R^N$  the conditions  $A_k$  and  $B_k$  need only be required for subfamilies with at most  $N + 1$  sets; they are automatically satisfied otherwise. The geometric implications of these results are shown in Corollary 1, which shows that, if the family is acyclic and every subfamily with at most  $N + 1$  sets has a nonempty intersection, then the whole family has a nonempty intersection.

Building on this, Theorem 6 gives a necessary and sufficient condition for the acyclic (and therefore nonempty) intersection of every subfamily of a family of finitely many sets in a general topological space  $X$ ; the union of every subfamily must be acyclic. Furthermore, if the family of sets is in  $R^N$ , the acyclicity is required only for subfamilies with no more than  $N + 1$  sets. For acyclic families, Corollary 2 gives a simple, necessary, and sufficient condition for the nonempty intersection of the whole family; particularly, the family has nonempty intersection if and only if every subfamily has an acyclic union. This result is just what is needed for the economic applications presented in §§3 and 4, as seen in Theorem 11 in §7.

So far we have considered families which have either empty or acyclic intersection and have excluded those where the sets have nonempty intersection, but this intersection is not acyclic. In several applications, for example, for non-convex economies it is necessary to consider the condition of limited arbitrage (5) which requires nonempty intersection, even when this intersection fails to be acyclic. Here Mayer-Vietoris is no longer useful, and other arguments are needed. The rest of this section extends the results to families which may have nonacyclic as well as nonempty intersection. This is achieved as follows: Theorems 7 and 8 establish an isomorphism between the homology of a space  $X$  and that of the nerve of a *simple* and of a *regular* cover respectively, as defined in §2. These include covers by sets which may be neither open, convex, acyclic, or even connected. Using this isomorphism, Theorems 9 and 10 prove necessary and sufficient conditions for nonempty intersection; these are similar to Theorem 6, but they are valid for simple and for regular families respectively.

Unless otherwise stated, the following results apply to a general topological space  $X$ , and the family  $\{U_\alpha\}_{\alpha \in S}$  satisfies

$$(8) \quad \bigcup_{\alpha \in S} U_\alpha = \bigcup_{\alpha \in S} (\text{int}_{U^S} (U_\alpha)),$$

where  $\text{int}_{U^S}(U_\alpha)$  denotes the interior of the set  $U_\alpha$  relative to the set  $U^S = \bigcup_{\beta \in S} (U_\beta)$ . A family satisfying this property (8) is called an *excisive family*. Since we can take  $X = \bigcup_{\alpha \in S} U_\alpha$ , (8) is a rather general specification. For

example, (8) is satisfied when the family consists of sets  $U_i \subset X$ , each of which is open in  $X$ . Note, however, that condition (8) does not require that the sets  $U_i$  be open in  $\bigcup_{\alpha \in S} (U_\alpha)$ . In fact, (8) is strictly weaker than the requirement that the sets  $U_i$  be open in  $X$ ; it includes, for example, families consisting of two closed sets  $C_1$  and  $C_2$  in  $R^n$  with  $C_1 \subset C_2$ . The role of (8) is to ensure the union and the intersection of any subfamily of  $\{U_\alpha\}_{\alpha \in S}$  define an excisive couple so that the Mayer-Vietoris sequence of reduced singular theory is exact (see Spanier [36, Theorems 3, 4 and Corollary 5, pp. 188-189]). An example in [36, p. 188] exhibits two closed path-connected sets  $Y_1$  and  $Y_2$  in  $R^2$  such that  $Y_1 \cup Y_2 = R^2$  which do not satisfy (8) and for which the corresponding singular Mayer-Vietoris sequence is not exact. Condition (8) prevents such pathologies.

**Theorem 5.** *An excisive family  $\{U_\alpha\}_{\alpha \in S}$  in  $X$  satisfies  $A_k$  if and only if it satisfies  $B_k$ .*

*Proof.* The first step in the proof is to establish the following duality result:

**Proposition 1.** *Consider an excisive family of sets in  $X$ ,  $\{U_\alpha\}_{\alpha \in S}$ , satisfying  $A_{k-1}$ , for  $k \geq 1$ . Then if  $\theta \subset S$  has  $k+1$  elements, for all  $q$*

$$(9) \quad H_q(U^\theta) \simeq H_{q-k}(U_\theta).$$

*Proof.* We proceed by induction. When  $k=1$ , the family has two sets, and this is the Mayer-Vietoris sequence for reduced singular homology as defined in §2. Assume the result is true for every family  $\{U_\alpha\}_{\alpha \in \theta}$  where  $\theta$  has  $k$  elements. Consider now a family  $\{U_\alpha\}_{\alpha \in \tau}$  of  $k+1$  elements satisfying  $A_{k-1}$ . Define  $\theta$  so that  $\tau = \{0\} \cup \theta$ , and  $V_\alpha = U_0 \cup U_\alpha$ ,  $\alpha \in \theta$ . The new family  $\{V_\alpha\}_{\alpha \in \theta}$  has  $k$  elements, and it satisfies  $A_{k-2}$  because the family  $\{U_\alpha\}_{\alpha \in \tau}$  satisfies  $A_{k-1}$  and by Mayer-Vietoris. Then

$$\begin{aligned} H_q(U^\tau) &= H_q(V^\theta) = H_{q-(k-1)}(V_\theta) \quad \text{by the induction hypothesis} \\ &= H_{q-k+1}(U_0 \cup [U_1 \cap \cdots \cap U_k]) \\ &= H_{q-k}(U_0 \cap [U_1 \cap \cdots \cap U_k]) \quad \text{by Mayer-Vietoris} \\ &= H_{q-k}(U_\tau), \end{aligned}$$

completing the proof of the proposition. The rest of the proof of Theorem 5 follows from Proposition 1 by induction on  $k$ .  $\square$

**Proposition 2.** *Let  $\{U_\alpha\}_{\alpha \in S}$  be an excisive family in  $R^n$  satisfying  $A_n$ . Then  $\{U_\alpha\}$  also satisfies  $A_k$  and  $B_k$  for all  $k \geq n$ . In particular, the intersection of this family is always nonempty.*

*Proof.* This follows from Theorem 5 and Mayer-Vietoris, because  $H_i(U) = 0$  for  $i \geq n$  for an open set  $U \subset R^n$ .  $\square$

**Corollary 1.** *Let  $\{U_\alpha\}_{\alpha \in S}$  be an acyclic excisive family in  $R^n$  with at least  $n+1$  elements. If every subfamily with  $n+1$  elements has nonempty intersection, then the whole family has a nonempty intersection.*

*Proof.* This follows from Proposition 2 because  $A_n$  is satisfied by acyclicity.  $\square$

**Example 1.** The conditions of Proposition 2 and Corollary 1 cannot be relaxed. In general, the family must have finite cardinality. Consider, for example, the infinite family in  $R^1$   $\{U_i\}_{i=1,2,\dots}$ ,  $U_i = (i, \infty)$ . Every subfamily of  $\{U_i\}_{i=1,2,\dots}$

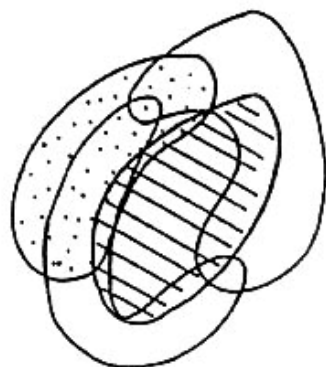


FIGURE 1

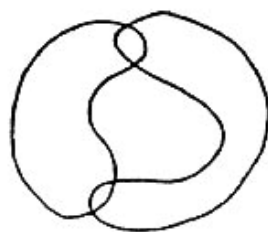


FIGURE 2

has acyclic union, but the whole family has empty intersection. Figure 1 shows that Corollary 1 does not hold for nonacyclic families; each three of these four sets in Figure 1 intersect, but the whole family has an empty intersection. Figure 2 also shows that Proposition 2 is not true when  $A_n$  is not satisfied. Here  $n = 2$ , and  $A_2$  is not satisfied because the union of two of the sets is not acyclic.

**Theorem 6.** *Let  $\{U_\alpha\}_{\alpha \in S}$  be an excisive family of  $k \geq 2$  sets. Then the intersection of every subfamily  $\bigcap_{\alpha \in \theta} U_\alpha$ ,  $\forall \theta \subset S$ , is acyclic (and hence nonempty) if and only if the union of every subfamily  $\bigcup_{\alpha \in \theta} U_\alpha$ ,  $\forall \theta \subset S$ , is acyclic; i.e., the family satisfies  $B_{k-1}$ . If the family  $\{U_\alpha\}_{\alpha \in S}$  is in  $R^n$ , then its intersection is acyclic if and only if its union  $\bigcup_{\alpha \in S} U_\alpha$  is acyclic and it satisfies  $B_j$  for  $j = \min(n, k - 2)$ .*

*Proof.* The first statement follows from Theorem 5. For the second statement, first let  $j = k - 2$ . Assume that  $\bigcup_{\alpha \in S} U_\alpha$  is acyclic and  $\{U_\alpha\}_{\alpha \in S}$  satisfies  $B_{k-2}$ . Then  $B_{k-1}$  is satisfied. By Theorem 5 so is  $A_{k-1}$  so that the intersection of the family is acyclic and thus nonempty. Reciprocally, if the intersection of the whole family is not empty, then  $A_{k-1}$  is satisfied and by Theorem 5 so is  $B_{k-1}$  so that the union of the family is acyclic. Now let  $j = n$ . By assumption and Theorem 5,  $A_n$  is satisfied. By Proposition 2 this implies that the whole family has nonempty intersection and that  $A_m$  is satisfied for all  $m \geq 0$ . Therefore by Theorem 5,  $B_m$  is satisfied for all  $m$ , and the family's union is acyclic.  $\square$

**Example 2.** Figures 2 and 3 show that the conditions of Theorem 6 cannot be relaxed. Figure 2 shows that "acyclic intersection" cannot be replaced by "nonempty intersection"; it depicts two sets which do intersect but have a nonacyclic union. Figure 3 shows that Theorem 6 is not true if we replace "acyclic union" by "contractible union" in its statement; it depicts two "comb" spaces having an acyclic (and hence nonempty) intersection, the point  $\{x\}$ . The union of the two comb spaces is acyclic, confirming Theorem 6, but it is not contractible.

**Corollary 2.** *An acyclic excisive family  $\{U_\alpha\}_{\alpha \in S}$  has nonempty intersection if and only if  $\forall \theta \subset S$ , the union of the subfamily  $\{U_i\}_{i \in \theta}$ ,  $\bigcup_{\alpha \in \theta} U_\alpha$ , is acyclic.*

*Proof.* This follows from Theorem 6 and the definition of acyclic families.  $\square$

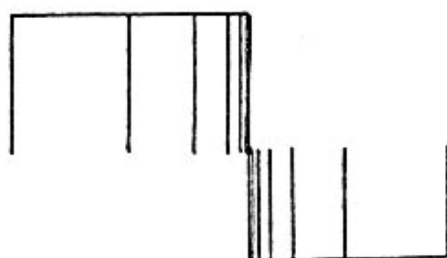


FIGURE 3

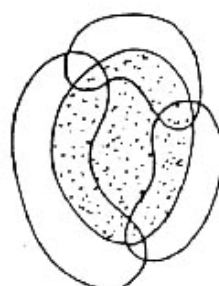


FIGURE 4

**Example 3.** The conditions of Corollary 2 cannot be relaxed. Figure 4 depicts a family of  $k = 4$  sets in  $R^2$  which does not satisfy  $B_2$  (or  $A_2$ ) because three of them do not intersect. The union of the family is acyclic, but the intersection is empty.

Until now we considered families which had either empty or acyclic intersection. The following results apply to *simple* and *regular* families, as defined in §2. These may consist, for example, of sets in  $R^n$  which are neither open nor acyclic or even connected. The families may have *nonacyclic, nonempty intersection*. Mayer-Vietoris is not useful in this context, and we must adopt a different approach.

If  $X$  is a simplicial complex, the expression  $X = \text{nerve } \{U_\alpha\}_{\alpha \in S}$  is used to indicate that  $X$  and nerve  $\{U_\alpha\}_{\alpha \in S}$  have the same combinatorial structure.

**Theorem 7.** Let  $\{U_\alpha\}_{\alpha \in S}$  be a simple cover of a simplicial complex  $X$  with set of vertices equal to  $S$ . Then  $X = \{U_\alpha\}_{\alpha \in S}$ .

*Proof.* The proof follows by induction on the number of sets  $k$ . Let the set of vertices  $S$  consist of  $k = 2$  elements. Then  $X$  is either a segment or a set of two points; assume  $X$  is a segment. Consider  $x \in \partial U_1$ . Since  $x \notin U_1$ ,  $x \in U_2$ . Therefore,  $\exists y \in U_1 \cap U_2$ . Now let  $X = \{x_1\} \cup \{x_2\}$ . Since  $U_\alpha \subset \text{star}(\alpha)$ ,  $U_1 \cap U_2$  is empty. Consider now the following inductive assumption for a set of vertices  $S$  of  $k + 1$  elements: the nerve  $\{U_\alpha\} = X$ , and if the  $k$  sets  $\{U_\alpha\}_{1 \leq \alpha \leq k}$  intersect, then  $\exists$  a simple family  $\{W_\alpha\}_{1 \leq \alpha \leq k+1}$  covering  $X$  with  $W_\alpha \subset U_\alpha \forall \alpha$  and an  $x \in \partial W_1 \cap \dots \cap \partial W_k$ . Now let  $S$  have  $k + 2$  sets. Assume  $X$  is a  $k + 1$  simplex. By the inductive hypothesis every subfamily of  $k + 1$  sets in  $\{U_\alpha\}$  intersects, and in particular,  $\exists$  a simple family  $\{W_\alpha\}_{1 \leq \alpha \leq k+1}$  with  $x \in \bigcap_{1 \leq \alpha \leq k} \partial W_\alpha$ . Let  $Z_{k+1} = W_{k+1} - I_x$ , where  $I_x$  is a closed segment in  $W_1 \cap \dots \cap W_k$ , and  $x \in \partial I_x$ . Take  $Z_{k+1}$  to be an element of the simple family  $\{Z_\alpha\}_{1 \leq \alpha \leq k+2}$  defined otherwise by  $Z_\alpha = W_\alpha$  for  $\alpha \leq k$  and  $Z_{k+2} = U_{k+2}$ . Then  $\forall \alpha$ ,  $Z_\alpha \subset U_\alpha$ ,  $\{Z_\alpha\}_{1 \leq \alpha \leq k+2}$  covers  $X$ , and  $x \in \partial Z_1 \cap \dots \cap \partial Z_{k+1}$ , so  $x \in Z_{k+2}$ . Therefore,  $x \in \bigcap_{1 \leq \alpha \leq k+2} Z_\alpha \subset \bigcap_{1 \leq \alpha \leq k+2} U_\alpha \neq \emptyset$ . Finally, if  $X$  is not a simplex,  $\bigcap_{1 \leq \alpha \leq k+2} U_\alpha = \emptyset$ , since  $U_\alpha \subset \text{star}(\alpha)$  for all  $\alpha \in S$ .  $\square$

The following result uses the definition of regular covers given in §2.

**Theorem 8.** Let  $\{C_\alpha\}_{\alpha \in S}$  be a regular cover of a simplicial complex  $X$ . Then nerve  $\{C_\alpha\}_{\alpha \in S} = X$ .

*Proof.* First we prove that Theorem 7 implies that if  $\{C_\alpha\}_{\alpha \in S}$  is a regular cover of  $X$ , then  $\bigcap_{\alpha \in S} C_\alpha \neq \emptyset$ . Let  $D_\alpha = C_\alpha \cap \text{star}(\alpha)$ ; then  $\bigcup_{\alpha \in S} D_\alpha = X$ . Now by Theorem 7

(10)

if  $\{U_\alpha\}_{\alpha \in S}$  is a simple family covering  $X$  with  $U_\alpha \supset D_\alpha$  for all  $\alpha$ ,  $\bigcap_{\alpha \in S} U_\alpha \neq \emptyset$ .

We now use (10) to prove  $\bigcap_{\alpha \in S} D_\alpha \neq \emptyset$ , by induction on  $k$ .

Case  $k = 1$ . If  $\bigcap_{\alpha=1,2} D_\alpha = \emptyset$ , then  $\exists U_1, U_2$  defining a simple family with  $\bigcap_{\alpha=1,2} U_\alpha = \emptyset$ , contradicting (10). Now let  $S$  have  $k + 1$  elements: by the inductive assumption,  $\bigcap_{1 \leq \alpha \leq k} C_\alpha \neq \emptyset$ . If  $[\bigcap_{1 \leq \alpha \leq k} C_\alpha] \cap C_{k+1} = \emptyset$ , then  $\exists$  a simple family  $\{U_\alpha\}$  s.t.  $[\bigcap_{1 \leq \alpha \leq k} U_\alpha] \cap U_{k+1} = \emptyset$ , contradicting (10). Thus  $\bigcap_{\alpha \in S} D_\alpha \neq \emptyset$  so that  $\bigcap_{\alpha \in S} C_\alpha \neq \emptyset$ .

Having established the result for the case where  $X$  is a simplex, the rest of the proof follows the proof of Theorem 7 by considering the family defined by the complements of the sets  $\{C_\alpha\}_{\alpha \in S}$  in  $X$ .  $\square$

The two following theorems extend the results of Theorem 6 to the cases of *simple* and *regular* families as defined in §2; here we are concerned with the nonempty intersection of the family, whether or not this intersection is acyclic.

**Theorem 9.** *Let  $\{U_\alpha\}_{\alpha \in S}$  be a simple family of  $k$  sets, such that every subfamily with  $k - 1$  elements has a nonempty intersection. Then the whole family has a nonempty intersection if and only if its union  $\bigcup_{\alpha \in S} U_\alpha$  is acyclic. If  $k > n + 1$ , we need to require only that every family of  $n + 1$  sets has a nonempty intersection.*

*Proof.* By assumption the  $(k - 2)$ -skeleton of nerve  $\{U_\alpha\}_{\alpha \in S}$  is the boundary of a  $k - 1$  simplex. Let  $X = \{U_\alpha\}_{\alpha \in S}$ . By Theorem 7 nerve  $\{U_\alpha\}_{\alpha \in S} = X$ . Therefore, all sets in the family  $\{U_\alpha\}$  intersect if and only if its union  $X = \bigcup_{\alpha \in S} U_\alpha$  is acyclic.  $\square$

**Theorem 10.** *Let  $\{C_\alpha\}_{\alpha \in S}$  be a family of  $k$  closed sets with  $[\alpha]_{\alpha \in \sigma} \subset \bigcup_{\alpha \in \sigma} C_\alpha$ , and  $\bigcap_{\alpha \in \sigma} C_\alpha \neq \emptyset$  for every subset  $\sigma$  of  $S$  with  $k - 1$  elements. Then  $\bigcap_{\alpha \in S} C_\alpha \neq \emptyset$  if and only if  $\bigcup_{\alpha \in \sigma} C_\alpha$  is acyclic. If  $k > n + 1$ , we need to require only that every family of  $n + 1$  sets has a nonempty intersection.*

*Proof.* This follows from the proof of Theorem 9, replacing Theorem 7 in the proof by Theorem 8.  $\square$

## 6. EXTENSIONS OF THEOREMS OF HELLY, CARATHEODORY, KKM, BROUWER, AND LERAY

The question of when sets intersect was studied in the classic theorems of Helly [27, 28] and of Knaster-Kuratowski-Marzukiewicz in [6]. They provided conditions which are sufficient for a family of sets in  $R^n$  to have a nonempty intersection, but their results are restricted to families with  $n + 1$  or more sets in the case of Helly's theorem and to families with exactly  $n + 1$  sets in the case of KKM's theorem, in both cases having either a convex structure or other particular characteristics. These two results are quite specific to the problems they study and appear to be different from each other. However, the problem of nonempty intersection in its general form has a clear geometrical structure

and can be dealt with by using topological tools. We showed in §5 that no restrictions on the number of sets is required, nor is convexity, acyclicity, or even connectedness of the sets. Furthermore, the families need not be in  $R^n$  or in any linear space. Once this is understood, the two classic results of Helly and KKM appear as special cases of our results. Brouwer's theorem is also a special case of our results, since it is known to be implied by the KKM theorem, as is Caratheodory's theorem, which follows from the Helly's theorem.

Helly's theorem is connected here to the Brouwer's fixed point theorem and to an extension provided here of Leray's theorem on acyclic covers. Our extension of Leray's theorem (Leray [31], Dowker [22], Cartan [7]) is in Theorems 7 and 8 of §5; while Leray's theorem applies to acyclic covers and proves the isomorphism of the homology of the nerve of the cover and that of the union of the family, our Theorems 7 and 8 significantly extend this result for covers consisting of sets which may not be acyclic nor open or even connected.

This section therefore exhibits how the results in §5 extend and unify several classical theorems. Proposition 2 in §5 extends Helly's striking theorem on the nonempty intersection of families in  $R^n$  having more than  $n + 1$  sets (Helly [27], Alexandroff and Hopf [1]) to possibly nonconvex and nonacyclic families with any number of sets in a general topological space  $X$ . Corollary 3 below is Helly's theorem. Since Helly's theorem implies Caratheodory's representation theorem (Eggleston [24]), Proposition 2 in §5 extends also Caratheodory's theorem to the same wide range of families. Corollary 4 is the Knaster-Kuratowski-Marzukiewicz theorem (Berge [6, p. 173], which follows immediately as a very special case of our Theorem 7 in §5. KKM's theorem is restricted to families of sets in  $R^n$  which cover an  $n$ -simplex, while our Theorem 7 applies to families in a general topological space of any cardinality, which cover any simplicial complex. An additional extension of the KKM is Corollary 5, which applies to simple families. Corollary 6 is the Brouwer fixed point theorem (Hirsch [29]). These results exhibit a common topological root for these classical and somewhat disparate results.

**Corollary 3** (Helly's theorem). *Let  $\{U_\alpha\}_{\alpha \in S}$  be a family of convex sets in  $R^n$  with at least  $n + 1$  elements. Then if every subfamily with  $n + 1$  sets has a nonempty intersection, the whole family has a nonempty intersection.*

*Proof.* This follows directly from Proposition 2 in §5, which is valid in much more generality for any number of sets in a general topological space, because convex sets define an excisive family.  $\square$

The following corollary requires no convexity:

**Corollary 4** (KKM Theorem). *Let  $\{C_\alpha\}_{\alpha \in S}$  be a regular cover of a  $k$ -simplex  $X$  as defined in §2. Then  $\bigcap_{\alpha \in S} C_\alpha \neq \emptyset$ .*

*Proof.* This follows directly from Theorem 8, which is valid more generally for any simplicial complex. Since nerve  $\{C_\alpha\}_{\alpha \in S}$  and  $X$  have the same combinatorial structure, it follows, in particular, that  $\bigcap_{\alpha \in S} C_\alpha \neq \emptyset$ .  $\square$

In addition, the following result extends the KKM theorem to a different class of covers, *simple covers*, as defined in §2, which need not satisfy any of the conditions of KKM theorem:

**Corollary 5** (Extension of KKM to simple families). *Let  $\{U_\alpha\}_{\alpha \in S}$  be a simple cover of a  $k$ -dimensional simplex  $X$ . Then  $\bigcap_{\alpha \in S} U_\alpha$  is not empty.*

*Proof.* This follows directly from Theorem 7, which is also valid for covers of any complex  $X$ .  $\square$

Since the KKM theorem follows directly from Theorem 8 as shown in Corollary 4, by presenting for completeness a well-known argument, we show that Brouwer's fixed point theorem also follows as an immediate corollary of our Theorem 8.

**Corollary 6** (Brouwer's fixed point theorem). *Let  $X$  be a  $k$ -simplex, and  $f : X \rightarrow X$  a continuous function. Then  $\exists x \in X : f(x) = x$ .*

*Proof.* The proof follows by contradiction. If  $f : X \rightarrow X$  has no fixed point, then it defines a retraction  $r : X \rightarrow \partial X$ . Let  $\partial X = \bigcup_i X_i$ , where  $X_i$  is the  $i$ th face of  $X$ , a  $k-1$  simplex. Now define the closed sets  $C_i = \{r^{-1}(X_i)\}$ ,  $i = 1, \dots, k+1$ . Then  $\{C_i\}_{i=1, \dots, k+1}$  is a closed cover of  $X$  satisfying the conditions of Corollary 4, so  $\bigcap_i C_i \neq \emptyset$ . But if  $p \in \bigcap_i C_i$ , then  $r(p) \in \bigcap X_i = \emptyset$ , a contradiction.  $\square$

## 7. MARKET EQUILIBRIUM AND SOCIAL CHOICE

Our final task is to establish the equivalence of the two economic problems, namely, the existence of a competitive equilibrium and the existence of a social choice function. A good way to start is to provide examples of spaces of preferences in order to illustrate the topological problem involved in social choice. By Theorem 3 in §4, this problem can be solved only for acyclic spaces of preferences.

A preference  $\rho$  is an ordering of the choice space  $R^n$  which is induced by a utility function  $u : R^n \rightarrow R$ , where we indicate  $x \succeq_\rho y \Leftrightarrow u(x) \geq u(y)$ . A smooth preference on  $R^n$  is defined by a smooth ( $C^2$ ) unit vector field  $\rho : R^n \rightarrow S^{n-1}$ , with the property that  $\exists$  a function  $u : R^n \rightarrow R$  such that  $\forall x \in R^n$ ,  $\exists \lambda(x) > 0$  such that  $\rho(x) = \lambda(x) Du(x)$ ; i.e., there exists a function  $u$  such that  $\forall x$ ,  $\rho(x)$  is collinear with the gradient of  $u$  (see Debreu [21]).

One example of a space of preferences  $P$  is the space of all smooth preferences on  $R^n$ , denoted  $\Gamma(R^n)$ , endowed with the sup norm,  $\|\rho - \kappa\| = \sup_{x \in R^n} \|\rho(x) - \kappa(x)\|$ . Another example of a space of preferences is the space  $P_L$  of all linear preferences on  $R^n$ , which are those preferences induced by linear utility functions on  $R^n$ ,  $n > 2$ . The space  $P_L$  is the sphere  $S^{n-1}$ . If the zero preference is also included, we have the space  $P_{LN}$  of all linear preferences on  $R^n$ —this space is  $S^{n-1} \cup \{0\}$ . Different preference spaces arise in different applications (for examples, see, e.g., Heal [26]). Typically, preference spaces are not linear nor convex or acyclic; for example, the space of smooth preferences  $\Gamma(R^n)$  is not acyclic [8].

Our last task is to establish the connection between the existence of a market equilibrium and the existence of a social choice function. Both problems depend on the characteristics of the traders' preferences, but they do so in two different ways. The market  $E$  has a finite set of preferences, one for each trader,  $\{\rho_1, \dots, \rho_H\}$ . The set of preferences in the economy is therefore a discrete finite set of points in the space of smooth preferences  $\Gamma(R^n)$  defined above. The social choice function, by contrast, is generally defined on large



spaces describing a universe of all possible preferences, typically a connected subset  $P$  of the space of all smooth preferences in  $\Gamma(R^n)$ , which is not a finite set.

In order to exhibit the connection between the two problems—the existence of market equilibrium and that of a social choice function—we define a space consisting of preferences which are naturally “close” to those of the preferences of the traders in the economy  $E$ . The space of preferences  $P_E$  consists of a large number of preferences, assumed to be a connected subspace of  $\Gamma(R^n)$ , all of which are, in a well-defined sense, similar to the preferences in the market  $E$ . We therefore need to define what is meant by a smooth preference which is similar to the preferences of the traders in the market  $E$ .

A smooth preference  $\rho \in P$  defined over allocations in  $R^{nH}$  is called *similar* to the preference of trader  $i \in E$  in position  $j$  when  $\forall x \in R^{nH}$ , the projection of  $\rho(x)$  on the  $j$ th copy of  $R^n$  is in the market cone of trader  $i$ ; i.e.,  $\forall x \in R^{nH}$ ,  $\rho^j(x) \in D_i$ . The interpretation of this condition is that the preference  $\rho$  increases in the direction of that of the trader  $i$  in position  $j$  for large utility values. The space  $P(E_\theta)$  of preferences similar to those of a subset  $\theta \subset \{1, \dots, H\}$  of traders in  $E$  was already defined in §4; it consists of all those smooth preferences  $\rho \in \Gamma(R^{nH})$  such that  $\forall x \in R^{nH}$ ,  $\rho^j(x) \in \bigcup_{i \in \theta} D_i$ . If we consider the problem of finding a social choice function for the space of preferences  $P(E_\theta)$  which are similar to those of some subset  $\theta$  of traders in the economy  $E$ ,  $\theta \subset \{1, \dots, H\}$ , then by Theorem 4 in §4 the necessary and sufficient condition is the acyclicity of  $\bigcup_{i \in \theta} D_i$ . The existence of a social choice function for every such space of preferences  $P(E_\theta)$ ,  $\forall \theta \subset \{1, \dots, H\}$  therefore requires

$$\forall \theta \subset \{1, \dots, H\}, \quad \theta \neq \emptyset \Rightarrow \bigcup_{i \in \theta} D_i \text{ is acyclic.}$$

Note that in order to solve the social choice problem we must go back to the properties of the family of market cones  $\{D_i\}_{i \in \{1, \dots, H\}}$  of the economy  $E$  defined in (4)—the same family of cones which define the condition of limited arbitrage (5).

Theorem 11 exhibits the identity between the problems of existence of a competitive equilibrium for a market  $E$  and the existence of a social choice function. Let  $E$  be a market as defined in §3. A *subeconomy*  $E_\theta$  of  $E$  is the market consisting of the those traders in  $E$  who belong to the set  $\theta \subset \{1, \dots, H\}$ , i.e.,

$$E_\theta = \{R^n, \Omega_i, u_i, i \in \theta\}.$$

**Theorem 11.** *The following properties of the economy  $E = \{R^n, \Omega_i, \rho_i, i = 1, \dots, H\}$  are equivalent:*

- (a)  $E$  has a competitive equilibrium.
- (b) Every subeconomy  $E_\theta$  of  $E$  has a competitive equilibrium.
- (c) Every subeconomy  $E_\theta$  of  $E$  with at most  $n + 1$  traders has a competitive equilibrium.
- (d) There exists a social choice function  $\Phi : P(E_\theta)^k \rightarrow P(E_\theta)$  satisfying conditions 4.2.1 and 4.2.2, for every space  $P(E_\theta)$  of preferences similar to those of the traders in a nonempty set  $\theta$ ,  $\forall \theta \subset \{1, \dots, H\}$ , and  $\forall k \geq 1$ .

*Proof.* The equivalence between (a) and (b) follows immediately from Theorem 1 in §3 and from the definition of limited arbitrage (LA) in (5). We establish next the equivalence of the statements (a) and (c). By Theorem 1,  $E$  has a competitive equilibrium if and only if  $E$  has limited arbitrage (LA) as defined in (5), i.e., if and only if the family of dual cones  $\{D_i\}_{i=1, \dots, H}$  has a nonempty intersection. Since  $\{D_i\}_{i=1, \dots, H}$  is an acyclic excisive family in  $R^n$ , by Corollary 1, (5) is true if and only if every subfamily of  $\{D_i\}_{i=1, \dots, H}$  with indices in a set  $\theta \subset \{1, \dots, H\}$  of at most  $n+1$  elements has nonempty intersection, i.e., if and only if the corresponding subeconomy  $E_\theta$  satisfies limited arbitrage (5), and therefore by Theorem 1 if and only if  $E_\theta$  has a competitive equilibrium.

The equivalence between statements (a) and (d) follows from Theorem 4 in §4 and from Theorem 6 and Corollary 2 in §5, because  $\{D_i\}_{i=1, \dots, H}$  is an acyclic excisive family, so

$$\bigcap_{i=1}^H D_i \neq \emptyset \Leftrightarrow \forall \text{ nonempty } \theta \subset \{1, \dots, H\}, \bigcup_{i \in \theta} D_i \text{ is acyclic. } \square$$

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#### REFERENCES

1. P. Alexandroff and H. Hopf, *Topologie*, Chelsea Publishing Co. Bronx, New York, 1965.
2. K. J. Arrow, *An extension of basic theorems of classical welfare economics*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (J. Neyman, ed.), Univ. of California Press, Berkeley, CA, 1951, pp. 507–532.
3. K. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, *Econometrica* 22 (1954), 264–290.
4. K. Arrow and F. Hahn, *General competitive analysis*, North-Holland, Amsterdam, New York, Oxford, and Tokyo, 1971 (fourth printing: 1986).
5. K. J. Arrow, *Social choice and individual values*. Cowles Foundation Monographs, Wiley, New York, 1953.
6. C. Berge, *Topological spaces*, Oliver and Boyd, Edinburgh and London, 1963.
7. H. Cartan, *Un theorem de finitude*, *Seminaire Cartan*, Expose 17, 1954.
8. G. Chichilnisky, *Social choice and the topology of spaces of preferences*, *Adv. Math.* 37 (1980), 165–166.
9. ———, *Intersecting families of sets: a topological characterization*, Working Paper no. 166, Univ. of Essex, February 1981.
10. ———, *The topological equivalence between the Pareto condition and the existence of dictators*, *J. Math. Econom.* 9 (1982), 223–233.
11. ———, *Topology and economics: the contribution of Stephen Smale*, From Topology to Computation, Proceedings of the Smalefest (M. Hirsch, J. Marsden, and M. Shub, eds.), Springer-Verlag, New York, 1993.
12. ———, *Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium*, Working Paper, Columbia Univ., 1992.
13. ———, *Markets, arbitrage and social choice*, Working Paper, Columbia Univ., 1991.
14. ———, *On strategic control*, *Quart. J. Econom.*, February 1993, 285–290.
15. G. Chichilnisky and G. M. Heal, *Necessary and sufficient conditions for a resolution of the social choice paradox*, *J. Econom. Theory* 31 (1983), 68–87.
16. ———, *Patterns of power*, *J. Public Econom.* 23 (1984), 333–349.

17. ———, *Existence of a competitive equilibrium in Sobolev spaces, without bounds on short sales*, *J. Econom. Theory* **59** (1993), 364–384.
18. G. Chichilnisky, *Market cones and the global convergence of a process of price adjustment*, Working Paper, Columbia Univ., 1993.
19. L. Danzer, V. Klee, and B. Grünbaum, *Helly's Theorem and its relatives*, *Proc. Sympos. Pure Math.*, vol. VII, Amer. Math. Soc., Providence, RI, 1963.
20. G. Choquet, *Lectures on analysis, Volume II, Representation Theory*, (J. Marsden, T. Lance, and S. Gerlhart, eds.), W. A. Benjamin, New York and Amsterdam, 1969.
21. G. Debreu, *Smooth preferences*, *Econometrica* **40** (1971), 603–615.
22. C. H. Dowker, *Lectures on sheaf theory*, Tata Institute of Fundamental Research, Bombay, 1957.
23. C. Eaves, *Homotopies for computation of fixed points*, *Math. Programming* **3** (1972).
24. H. G. Eggleston, *Convexity*, Cambridge Tracts in Math. Math. Phys., **47**, Cambridge Univ. Press, Cambridge and New York, 1958.
25. R. Guesnerie and C. Oudiz, *On economic games which are not necessarily superadditive*, *Econom. Lett.* (1979).
26. G. Heal, *Contractibility and public decision making*, *Social Choice and Welfare* (P. Pattanaik and M. Salles, eds.), chapter 7, North-Holland, Amsterdam, 1983.
27. E. Helly, *Über Mengen Konvexen Körper mit Gemeinchaftlichen Punkten*, *Iber. Deutch. Math. Verein* **32** (1933), 175–186.
28. ———, *Über Systeme Abgeschlossener Mengen mit Gemeinchaftlichen Punkten*, *Monatsch. Math.* **37** (1930), 281–302.
29. M. Hirsch, *A proof of the non-retractability of a cell onto its boundary*, *Proc. Amer. Math. Soc.* **14** (1963), 364–365.
30. M. Hirsch and S. Smale, *On algorithms for solving  $f(x) = 0$* , *Comm. Pure Appl. Math.* **32** (1979), 281–312.
31. J. Leray, *Sur la forme des espaces topologiques, et sur les point fixes des representations*, *J. Math. Pures Appl.* (9) **24** (1945), 95–248.
32. J. F. Nash, *Equilibrium points in  $n$ -person games*, *Proc. Nat. Acad. Sci. U.S.A.* **36** (1950), 48–49.
33. H. Scarf, *The core of an  $n$ -person game*, *Econometrica* **35** (1967), 50–69.
34. S. Smale, *Exchange processes with price adjustment*, *J. Math. Econom.* **3** (1976), 211–226.
35. ———, *A convergent process of price adjustment and global Newton method*, *J. Math. Econom.* **3** (1976), 107–120.
36. E. H. Spanier, *Algebraic topology*, Springer-Verlag, New York, Heidelberg, and Berlin, 1966.
37. J. Von Neumann, *A model of general economic equilibrium* (G. Morgenstern, transl.), *Review Econom. Stud.* **XIII** (1945–1946), 1–9.

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