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Nonlinear Functional Analysis and Optimal Economic Growth*

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A problem of existence and characterization of solutions of optimal growth models in many sector economies is studied. The social utility to be optimized is a generalized form of a preference depending additively on consumption at the different dates of the planning period. The optimization is restricted to a set of admissible growth paths defined by production–investment–consumption relations described by a system of differential equations. Sufficient conditions are given for existence of a solution in a Hilbert space of paths, without convexity assumptions on either the utilities or the technology, using techniques of nonlinear functional analysis. A characterization is given of the utilities which are continuous with respect to the Hilbert space norm. Under convexity assumptions a characterization is also given of optimal and efficient solutions by competitive prices.

1. INTRODUCTION AND MOTIVATION

The goal of this paper is to study certain problems of nonlinear functional analysis and their applications to intertemporal allocation policies available to an economy which chooses, at each point in time, between consumption and investment in different productive sectors. We start with an informal description of the problem; for further references see, for instance, [1].

We assume that there are initially given values of capital stocks in the n sectors of the economy, denoted $K_i(0) \in R^+$, $i = 1,..., n$, and an initial total population denoted $L(0) \in R^+$. Each sector has a given production technology that uses as inputs capital and labor; the technology of the ith sector is described by a production function

$$F^i: R^{n+1} \to R,$$

where $F^i(L_i, K_{1i}, ..., K_{ni})$ represents the output of the ith sector, $L_i \in R$ denotes allocation of labor to sector i, and $K_{ji} \in R$ denotes the allocation of type j capital goods to sector i. When the inputs L_i and K_{ji} are positive, the output is assumed

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to be positive also. The total population, represented by the real variable L, is assumed to grow at an exponential rate $\mu \in R^+$,[1] i.e., at time t,

$$L(t) = L_0 e^{\mu t}.$$

The total population $L(t)$ constraints the total available supply of labor in all sectors of the economy through time, i.e., $\sum_{i=1}^{n} L_i(t) \leqslant L(t)$, $t \in [0, \infty)$. Similarly, allocation of type i capital goods to all sectors is constrained by the total amount of available capital goods of type i through time, denoted $K_i(t) \in R^+$, i.e., $\sum_{j=1}^{n} K_{ji}(t) \leqslant K_i(t)$, $t \in [0, \infty)$. Let $C_i(t)$ denote (instantaneous) consumption of the goods produced by the ith sector at time t, and let $\dot{K}_i(t) = (d/dt)\, K_i(t)$ denote the rate of change of capital stock or investment in sector i at time t. At each point in time, for each sector i it is assumed that the sum $C_i(t) + \dot{K}_i(t)$ (consumption of goods produced by the sector plus investment realized on that sector) cannot exceed the production capabilities of that sector's technology, taking into account that capital stocks depreciate at a (linear) rate $\gamma \in R^+$. Formally, this is represented by the inequalities

$$C_i(t) + \dot{K}_i(t) \leqslant F^i(L_i(t), K_{1i}(t),..., K_{ni}(t)) - \gamma K_i(t),$$

for

$$i = 1,..., n, \qquad t \in [0, \infty),$$

which give implicitly a choice between present and future consumption, provided F^i increases with K_{ji}.[2]

The problem is then to choose the instruments: consumption $C(t) = (C_1(t),..., C_n(t))$, types of investment $K(t) = (K_{ji}(t))$, $i = 1,..., n$, $j = 1,..., n$, and allocation of labor $L_i(t)$ as functions of time, so as to maximize a criterion function, which we discuss next.

The time dependent welfare or criterion function is described by

$$W(C) = \int_0^{\infty} e^{-\delta t} u(C(t)/L(t), t)\, dt, \tag{1}$$

where, as described above, $C(t) = (C_1(t),..., C_n(t))$ is the vector of current consumption level of the n types of commodities at date t, and $L(t)$ represents the population at date t. u is a given real valued function on R^{n+1} representing the (instantaneous) social utility derived from per capita consumption $C(t)/L(t)$ at

[1] The pattern of growth of the population can be given different forms, or it can be determined in part endogenously, see, for instance [9]. Given the fact that in this model the population growth is exponential and the welfare function W depends on per capita consumption, the rate of growth of the population μ is later embodied in the parameter β (see (1) and the statement of problem (P) below).

[2] If less is consumed at time t, more can be invested at time t and thus more could be produced and consumed in the future.

time t, and $\delta \in (0, 1)$ is a social discount factor representing choice of substitution between future and present consumption. $W(C)$ is a generalized form of a typical discounted preference function which depends additively on consumption at the different dates of the planning period, as studied, for instance, by Koopmans [7]. $W(C)$ represents the present discounted social utility of stream of per capita consumption.

We now give a formal statement of the problem. From here on lower case letters shall represent per capita quantities, for instance, $k_{ji} = K_{ji}/L$, $c_i = C_i/L$, etc.; the dependence of c_i, k_{ji}, etc. on time will not always be indicated to simplify notation. In per capita form the model becomes

$$\text{(P)} \qquad \underset{c_1,\ldots,c_n}{\text{Maximize}}\; W(c_1, \ldots, c_n),$$

where

$$W(c_1, \ldots, c_n) = \int_0^\infty e^{-\delta t} u(c_1(t), \ldots, c_n(t), t)\, dt,$$

and $\delta \in (0, \infty)$. The real valued measurable functions $c_1, \ldots, c_n$ on which the maximization is performed are restricted to a region within a function space where conditions (a) and (b) below are satisfied, for some functions $k = (k_{ji})$ and $l = (l_i)$ $(i, j = 1, \ldots, n)$ in certain function spaces:[3]

$$c_i + \dot{k}_i \leqslant F^i(l_i, k_{1i}, \ldots, k_{ni}) - \beta k_i; \qquad i = 1, \ldots, n, \quad \beta \in R^+, \tag{a}$$

and

$$\sum_{j=1}^{n} k_{ji} = k_i, \qquad i = 1, \ldots, n;$$

$$\sum_{i=1}^{n} l_i = 1; \tag{b}$$

$$k_i(0) = k_{i0}, \qquad i = 1, \ldots, n.$$

The c_i's, k_{ji}'s, and l_i's which represent per capita consumption, capital stocks, and labor path are all assumed to be positive real valued functions on $[0, \infty)$; the $\dot{k}_i$'s which represent per capita rates of change of capital stocks or investment flows are not necessarily positive real valued functions on $[0, \infty)$. The equalities of (a) and (b) above reflect the assumption that all available resources are utilized, and that all that is not invested is consumed.

In this paper we give sufficient conditions for existence of an optimal solution to the above problem in certain function spaces without convexity assumptions

[3] It is assumed here that the production technology is a homogeneous function so that (a) can be written in per capita form; the results can be immediately extended to nonhomogeneous cases when the variable L of population is bounded by above, instead of growing at an exponential growth as assumed above. See also footnote 1.

on either the technology (F) or the welfare function (W). We also prove uniqueness of the solution and existence of competitive prices for the optimal solution under convexity assumptions; a characterization is also given of efficient paths by competitive prices. We shall briefly discuss the meaning of the terms involved and of the results in economic and mathematical terms. *Prices* in this context are functions which assign a positive numerical (present) value to each consumption path $c(t)$; this value is assumed to be a continuous linear real valued function, and thus prices are continuous linear functionals defined on the space of consumption paths (see, for instance, [2]). When the value in a given price system p of all consumption paths within a set S is maximized at a consumption path $\bar{c}(t)$, this path is called *competitive* in S with respect to the given price system $\bar{p}$, and $\bar{p}$ is called a *competitive price* for $\bar{c}(t)$.[4] A consumption path $c(t)$ is called *efficient* within a set if it is maximal in that set with respect to a given ordering in the space of paths. (See also footnote 7 and definitions in Sect. 2.)

The question of existence of a solution of problem (P) gives methodological validity to the economic model; it is of importance to be able to prove it for both convex and nonconvex technologies (F) and utilities (u) (see Theorem 1). Nonconvex technologies appear typically in productive sectors which exhibit increasing returns to scale for some values of their inputs, such as certain public services: communications, energy, etc. The question of existence of competitive prices for the optimal or efficient growth paths addresses a decentralization problem in economic theory. Mathematically, the question is that of the existence of an adequate continuous linear functional separating the set of feasible paths from a translation of the positive cone with vertex on the feasible element c^* which optimizes the function W. Such a function will define a competitive price system p^* for c^*, and if any other path is strictly larger than c^* (in the order of the function space) then the value in p^* will be strictly larger. The question of existence of such a price system in economic terms is: Under what conditions will there exist a price p^* such that the (decentralized market mechanism of) competitive (value maximizing) allocation of resources in such a price system will drive the economy towards a growth path which is optimal from the social welfare utility W (centralized) viewpoint? The existence of such prices is proven here under convexity assumptions on the technology and the social utility function u. The method of the proof indicates how such prices may fail to exist in nonconvex cases.

We next discuss certain problems appearing in the choice of appropriate function spaces for the study of problem (P). By definition, the spaces of prices are the duals of the spaces of consumption paths on which the optimization is

[4] Competitive prices for optimal paths can be alternatively defined as prices satisfying an "intertemporal profit maximization" condition, which is well defined only in discrete time models, i.e., when the variable takes integer values. It can be shown that these two definitions are equivalent when the model is translated to a discrete time in our context, since the spaces of consumption paths and of prices are both (dual) Hilbert spaces.

performed. If the space of consumption paths was given, for instance, a sup norm, making it an L_∞ space, certain prices given by purely finitely additive elements in L_∞^* would have no natural economic interpretation, since they may assign a nonzero value to a whole consumption path c, while assigning for each t, zero value to the current consumption at time t, $c(t)$.[5] For this, among other reasons, L_p spaces with $1 \leqslant p < \infty$, and especially Hilbert spaces with L_2 norms, become natural candidates for spaces of consumption paths. However, the norms of these spaces have certain disadvantages with respect to the sup norm, which has biased in previous works in the area the choice of consumption path spaces in favor of L_∞ with the accompanying difficulties produced by the lack of reflexivity [2]. One inconvenient feature of L_2 spaces is that since their topology is weaker, it is harder to show conditions on u which yield L_2-continuity of nonlinear functionals such as W. Another is that in economic theory the admissible sets of paths on which the optimization is performed are usually assumed to be contained in positive cones: all L_p spaces with $1 \leqslant p < \infty$ have natural positive cones[6] with no interior or internal points while a basic tool needed to prove existence of competitive prices for optimal or efficient programs,[7] the Hahn–Banach theorem, requires one of the convex sets being separated to have an interior or an internal point [5],[8] so that these tools do not apply here. However, if the objective function being maximized is shown to be continuous in a weaker L_2 topology, one can overcome this problem.[9] Thus, the question of existence of prices is also related to the existence of appropriate continuous functionals, if one is to work on L_2. In Proposition 1, we give an extension of a result of continuity of certain nonlinear operators on L_p spaces of Krasnoselskii [8] which provides a characterization of continuous discounted additive time-dependent nonlinear utilities of the form of W, defined on an L_2

[5] This occurs, for instance, when the function part of a price p given by a purely finitely additive measure on $L_\infty[0, \infty)$ is zero for all $t \in [0, \infty)$, while p as a functional on L_∞ is not identically zero [5].

[6] An element f of $L_p(R, \mu)$ is called positive if $f(t) \geqslant 0$ for μ-almost all t in R.

[7] An element c of $L_p(R, \mu)$ is called *efficient* in a subset S if the translation of the positive cone with vertex c intersects S only at c.

[8] The hypothesis that one of the convex sets being separated contains an interior point can be weakened to the assumption that one of the sets has an *internal point* (see [5]) relative to the least closed vector subspace containing the set; this hypothesis, which is not satisfied in our model, cannot be eliminated, for a counterexample, see [3]. Dieudonné [4] also shows that, in a nonreflexive space, such as L_∞, two closed convex bounded sets without a common point may not have any closed separating hyperplane. If the space is reflexive (e.g., L_2) such sets can be separated by a closed hyperplane. In our problem, however, the two closed sets do have one point in common, namely the efficient or optimal path, so this result also does not apply, and new tools have to be used here.

[9] Basically, one shows that in this case one of the sets being separated is contained in a convex set which is the inverse image of an open set under a continuous map and is still disjoint from the other set being separated. (See Theorem 2 of Sect. 2.)

space of consumption paths. This characterization is given here on the economic parameters of W: the social utility function u, and the discount factor δ.

The results we obtain by use of Hilbert space techniques for existence and characterization of solutions, and for characterization of continuous nonlinear functions are new to the literature. In related models results in this direction were obtained with different techniques, for instance, in one sector convex models with discrete time, i.e., when the variable t takes integer values, and in continuous time models ($t \in [0, \infty)$), which have a more complex structure, related results were obtained mostly by the use of optimal control methods which require certain additional "transversality conditions" and also convexity for existence of solutions (see, for instance, [1]). Our results represent a strong extension of existent ones, and apply to a more general class of problems. They are obtained by using a different technique, which we now discuss. The weighted L_2 space of admissible consumption paths (with a finite measure on $[0, \infty)$) used here, denoted H_λ^0, contains as a dense subspace the space of bounded paths L_∞ which has been used in the economic literature [2]. In order to be able to work in L_2, given the relation between the feasible consumption and investment paths of (a) above, the space of admissible capital accumulation paths $k_i(t)$ is given a certain Hilbert space structure called a Sobolev space [13]. In view of Sobolev's inequality [13] the space of capital accumulation paths $k_i(t)$ is contained in C_0, i.e., the paths $k_i(t)$ are continuous, which is a desirable economic feature.

The topology of H_λ^0 is given by an L_2 norm denoted $\|\cdot\|_\lambda$, which is related to the discount factor δ in W. This topology is, of course, weaker than the sup norm, making it easier for sets to be compact but harder for functions to be continuous. However, by the results of Lemmas 1 and 2, Proposition 1, and Theorem 1, it represents a useful adaptation to the model: usual feasible consumption sets are $\|\cdot\|_\lambda$ compact and a wide class of welfare utility functions is $\|\cdot\|_\lambda$ continuous, which allows us to prove existence of solutions without convexity assumptions. Also, the notion of distance in this space seems quite well fitted for these types of discounted models. (See Proposition 1 and the following remark.)

Further advantages of these results of existence and characterizations of solutions in Hilbert spaces, which are not exploited here, include the use of tools for approximation of solutions by gradient methods which are available in Hilbert spaces for computing optimal paths of economic growth. Also, other nonlinear functional analysis techniques which are used in the study of differential operators defined on Sobolev's spaces (see, for instance, [10]) may now become useful for the study of the Euler–Lagrange differential equations associated ro problem (P), which can also be viewed as an integro-differential operator problem.

We proceed as follows:

In Lemmas 1 and 2, we establish compactness in H_λ^0 of the set of feasible consumption paths, under natural economic assumptions made on the technology of the model. This requires consideration of the relationship between different norm topologies on the set of feasible consumption and capital accumulation paths (defined in Sect. 2). Next, a necessary and sufficient condition for continuity of welfare functions W defined by the discounted sum of time-dependent utilities in the norm $\|\cdot\|_\lambda$ is given (Proposition 1). These results yield existence of a solution without requiring convexity assumptions on either the technology or the welfare function (Theorem 1). For cancave welfare functions and convex technologies, uniqueness follows (Theorem 1), and a version of the Hahn–Banach theorem and certain results on continuity of positive functionals are used to prove existence of positive competitive prices for the optimal solution (Theorem 2). A characterization is also given of efficient optimal paths in this model by strictly positive competitive prices in cases where the utility u is not necessarily increasing (Corollary 1).

2. Existence of Solutions

Let $L_\infty[0, \infty)$ be the space of essentially bounded, real valued functions on $[0, \infty)$, with the sup norm denoted $\|\cdot\|_\infty$. If f and g are in $L_\infty[0, \infty)$ define the inner product

$$(f, g)_\lambda = \int_0^\infty e^{-\lambda t} f(t) \cdot g(t)\, dt.$$

This inner product represents the discounted present value (at time 0) of the consumption plan f in price system g, with discount factor λ.[10]

Let

$$\|f\|_\lambda = (f, f)_\lambda^{1/2}$$

The completion of L_∞ under this norm is an L_2 space with the finite measure on $[0, \infty)$ given by the density function $e^{-\lambda t}$. We denote this space H_λ^0, and its L_2 norm $\|\cdot\|_\lambda$ to bring attention to the parameter λ in its definition. The relation between λ and the discount factor δ of the function W of problem (P) is studied in Proposition 1 below; in view of this proposition, although all spaces H_λ^0 are isomorphic for any λ, only certain values of λ are adequate for this model. Similarly, one defines H_λ^1:

Let f and g be C_b^1 functions (continuously differentiable and bounded) and define

$$(f, g)_\lambda^1 = \int_0^\infty e^{-\lambda t} \sum_{k=0}^{1} D^k f(t)\, D^k g(t)\, dt,$$

and

$$\|f\|_\lambda^1 = ((f, f)_\lambda^1)^{1/2}.$$

[10] By definition of discounted present value.

The completion of C_b^1 under the norm $\| \cdot \|_\lambda^1$ is denoted H_λ^1, and it is a Hilbert space which is called a (weighted) Sobolev space (see for instance, [10, 13]).

Let l denote the vector $(l_1, ..., l_n) \in R^n$, and m the matrix

$$\begin{pmatrix} k_{11} & \cdots & k_{n1} \\ \vdots & & \vdots \\ k_{1n} & \cdots & k_{nn} \end{pmatrix}.$$

Let k^i denote the vector $(k_{1i}, ..., k_{ni})$, and $k = (k_1, ..., k_n)$. Recall that k_i is defined to be $\sum_{j=1}^n k_{ji}$ ((b) of problem (P)).

Let $F(l, m)$ be a vector valued function $F(l, m)$: $R^{n+} \times R^{n^2+} \to R^{n+}$, given by

$$F(l, m) = (F^1(l_1, k_{11}, ..., k_{n1}), ..., F^n(l_n, k_{1n}, ..., k_{nn})),$$

where, for each i, $F^i(l_i, k^i)$: $R^{n+1+} \to R^+$ is a function representing the technology of sector i.

We make the following assumptions on the technology of the model:

A1. F admits an extension to a real valued function F_1 defined on a neighborhood of $R^{n+} \times R^{n^2}$, F_1 continuously differentiable and nondecreasing.

A2. There exists a vector $\bar{k} = (\bar{k}_1, ..., \bar{k}_n)$ such that

$$F^i(l, k^i) < \beta k_i, \qquad \text{for all } l \leqslant 1, \quad k_i \geqslant \bar{k}_i.$$

Furthermore, $k_{i0} < \bar{k}_i$ for all i where $\bar{k}^i = (\bar{k}_{1i}, ..., \bar{k}_{ni})$ and k_{i0} is $k_i(0)$ as given in (b) of problem (P). This assumption is basically a technological constraint on production: after certain levels of capital stocks the technology is constrainted in its per capita increases of productivity by the costs of maintenance of (per capita) capital stock, represented by the depreciation parameter β.

Let $C_{k_0} \times K_{k_0}$ be a set of all consumption paths in $(H_\lambda^0)^n$ and all paths of capital accumulation or allocations of capital to sectors in $(H_\lambda^1)^{n^2}$, defined by[11]

$$(c(t), m(t)) \in C_{k_0} \times K_{k_0}$$

when

(i) $c(t) \in L_\infty[0, \infty)^{n+}$, i.e., $c(t)$ is a measurable essentially bounded nonnegative path.

(ii) $m(t) \in H_\lambda^1[0, \infty)^{n^2+}$, and hence $k^i(t) \in H_\lambda^1[0, \infty)^n$, for all i. Note that by Sobolev inequality [13], $H_0^1[0, \infty) \subset C_0[0, \infty)$, i.e., the k_{ji}'s are continuous.

[11] $(H_\lambda^0)^n$ denotes the Cartesian product of H_λ^0 with itself n times. Similarly, for H_λ^1 and L_∞.

(iii) $k(0)$ (denoted also k_0) $= k_{ji}(0)$, with $k_{ji}(0)$ satisfying (b) of problem (P).

(iv) Constraints (a) and (b) are satisfied a.e. for each pair $(c(t), m(t))$ in $C_{k_0} \times K_{k_0}$ for some measurable path $l(t)$ in $[L_\infty[0, \infty)]^{n+}$,

(v) $0 \leqslant c(t) \leqslant F(l(t), m(t))$ a.e.[12], and

(vi) For $|h| < 1$, and all $t \in [0, \infty)$, there exists an $N > 0$ such that

$$| \Delta c(t, h)| \leqslant N \,| \Delta F(t, h)| ,$$

where $\Delta c(t, h) = c(t + h) - c(t)$, and $\Delta F(t, h)$ denotes a similar variation function of F.

Condition (vi) bounds changes in the feasible consumption paths in small intervals of time by a constant times changes in the variation of output. This condition is used in the proof of existence of an optimal path; it is not necessary for existence of solutions in discrete time models, i.e., when the variable t takes integer instead of real values.

C_{k_0} is called the set of *feasible consumption paths*, and K_{k_0} the set of all *feasible capital matrix paths*, with initial capital stock allocation k_0. The space of all feasible capital paths $k(t)$ corresponding to matrices in K_{k_0} is denoted G_{k_0}.

We also make the following assumption on the welfare function:

A3. $u: R^n \times R \to R^+$ is a strictly increasing function of c. It satisfies a Caratheodory condition, i.e., $u(c, t)$ is continuous with respect to $c \in R^n$ for almost all t in R, and measurable with respect to t for all values of c, (see [8]), and for all $(c, t) \in R^n \times R$

$$| u(c, t)| \leqslant b(t) + \alpha \,| c |^2,$$

where $b(t) \geqslant 0$, $\alpha \in R^+$ and $\int_0^\infty b(t)\, e^{-\lambda t} < \infty$.

Remark. By assumption A3 the function W of problem (P) is increasing in c, and hence it suffices to restrict the maximization problem to a region satisfying constraints (a) and (b) with an equality instead of an inequality in (a).

The next two lemmas show $\|\cdot\|_\lambda$ compactness of the sets of feasible consumption paths C_{k_0}; by the above remark, it suffices to show $\|\cdot\|_\lambda$ compactness of C_{k_0} when the constraint (a) of problem (P) is an equality.

LEMMA 1. *Under assumptions* A1 *and* A2 *for each initial capital stock allocation* k_0 *in* R^{n^2+} *the set* C_{k_0} *is a* $\|\cdot\|_\infty$ *bounded,* $\|\cdot\|_\lambda$ *closed subset of* $(H_\lambda^0)^{n+}$.

Proof. We first check uniform $\|\cdot\|_\infty$ boundedness of C_{k_0}. Since by (a) of problem (P), for any c in C_{k_0},

$$c(t) + \dot{k}(t) = F(l(t), m(t)) - \beta k(t) \qquad \text{a.e.} \tag{1}$$

[12] It can be shown that condition (v) can be replaced by the weaker condition: $c(t) \leqslant \int_{\max(0, t-M)}^t F(l(t), m(t))\, dt$, for some real number $M > 0$.

for some m in K_{k_0} and some admissible l, where βk represents the vector $(\beta k_1, ..., \beta k_n)$, and since by (v) of the definition of $C_{k_0} \times K_{k_0}$,

$$0 \leqslant c(t) \leqslant F(l(t), m(t)) \qquad \text{a.e.,} \tag{2}$$

it is enough to show that the set of values of $F(l(t), m(t))$ is bounded by above.

Note that by (1) and (2)

$$\dot{k} + \beta k = F(l, m) - c \geqslant 0 \qquad \text{a.e.,}$$

so that

$$\dot{k}(t) \geqslant -\beta k(t) \qquad \text{a.e.} \tag{3}$$

Also, by assumption, c is nonnegative, so that

$$\dot{k} \leqslant F(l, m) - \beta k \qquad \text{a.e.} \tag{4}$$

By Assumption A2 there exists a $\bar{k}$ in R^{n^2+} with $F^i(l, k^i) - \beta k_i < 0$

$$\text{for } k > \bar{k}, \text{ so that } \dot{k} \text{ is negative for } k > \bar{k}. \tag{5}$$

Thus, by (3), (4), and (5),

$$k_0 e^{-\beta t} \leqslant k \leqslant \bar{k}, \tag{6}$$

i.e., the admissible k's in G_{k_0} are bounded by above. Hence K_{k_0} is also bounded by above. Also, since by (b) of problem (P) the admissible l's are bounded by above in the sup norm, it follows that the right-hand side of (2) is bounded by continuity of F, which yields a uniform $\| \cdot \|_\infty$ bound for C_{k_0}.

We now show that C_{k_0} is $\| \cdot \|_\lambda$ closed. Let $\{c^\alpha\}$ be a net in C_{k_0}, converging in $\| \cdot \|_\lambda$ to a path c.

We shall see that there exists an m in k_0 such that (c, m) satisfies (a) and (b) of problem (P) for some admissible l. Let $m^\alpha \in K_{k_0}$ and $k^\alpha \in G_{k_0}$ be the capital paths corresponding to consumption path c^α in Eq. (a), which exist by definition of C_{k_0}. Note that, by (3) and (4) above, for all α,

$$-\beta k^\alpha \leqslant \dot{k}^\alpha \leqslant F(l^\alpha, m^\alpha) - \beta k^\alpha \qquad \text{a.e.,} \tag{7}$$

where l^α is a corresponding labor allocation path. Also, by (6) for all α

$$k_0 e^{-\beta t} \leqslant k^\alpha \leqslant \bar{k}. \tag{8}$$

By (7) and (8) the family $\{k^\alpha\}$ is equicontinuous and bounded at each point t and hence so is the family $\{m^\alpha\}$ by its definition. Thus, by a theorem of Ascoli [11, p. 179, Corollary 34], $\{m^\alpha\}$ contains a subsequence which converges pointwise to a continuous function, say, $m(t)$. By (7) and (8), both norms $\| m \|_\lambda$ and $\| m \|_\lambda^1$ are well defined, so that $m(t) \in H^{n^2+}$. Also, (c, m) satisfies (a) and (b) of problem (P) for some l, the limit of the corresponding l^α's, which is also bounded by (b) of problem (P). Thus the path $(c(t), m(t))$ is in $C_{k_0} \times K_{k_0}$, and thus C_{k_0} is closed. This completes the proof.

LEMMA 2. *C_{k_0} is precompact in the $\| \cdot \|_\lambda$ norm.*

Proof. It suffices to consider the case $n = 1$. Note that the set C_{k_0} is precompact in the $\| \cdot \|_\lambda$ norm if and only if the set

$$D = \{g: g(t) = c(t) \cdot e^{-\lambda t/2}, \text{ for } c \text{ in } C_{k_0}\}$$

is precompact in $L_2[0, \infty)$ with the Lebesgue measure. To prove precompactness of D we use the results of Lemma 1.

First, note that since, by Lemma 1, C_{k_0} is bounded in the $\| \cdot \|_\infty$ norm, say, by a constant C, then if $g \in D$

$$\int_0^\infty | g(t)|^2 \, dt = \int_0^\infty e^{-\lambda t} \, | c(t)|^2 \, dt \leqslant C \int_0^\infty e^{-\lambda t} < B$$

for some $B > 0$. Hence, D is bounded in the L_2 norm.

Next we need to verify, in view of [5, Theorem IV, Sect. 8.20], the following conditions (9) and (10):

$$\lim_{A \to \infty} \int_A^\infty e^{-\lambda t} \, | c(t)|^2 \, dt = 0 \tag{9}$$

uniformly for all c in C_{k_0}, and

$$\lim_{x \to 0} \int_0^\infty | e^{(-\lambda/2)(t+x)} c(t + x) - e^{(-\lambda t/2)} c(t)|^2 \, dt = 0 \tag{10}$$

uniformly for c in C_{k_0}.

Note that

$$\int_A^\infty e^{-\lambda t} c(t))^2 \, dt \leqslant \int_A^\infty e^{-\lambda t} C \, dt \tag{11}$$

for c in C_{k_0}, where C is the $\| \cdot \|_\infty$ bound of C_{k_0}. Since the right-hand side of (11) goes to zero independently of c as A goes to ∞, because $e^{-\lambda t}$ is in $L_2[0, \infty)$, then (9) is satisfied in our case.

Now, we prove that (10) is also satisfied. Let $| x | < 1$.

$$\int_0^\infty | e^{(-\lambda(t+x))/2} c(t + x) - e^{(-\lambda t/2)} c(t)|^2 \, dt$$

$$\leqslant \int_0^\infty (| e^{-\lambda(t+x)/2} c(t + x) - e^{(-\lambda(t+x))/2} c(t)| + | e^{(-\lambda(t+x))/2} c(t) - e^{(-\lambda t)/2} c(t)|)^2 \, dt \tag{12}$$

$$\leqslant \int_0^\infty (| e^{(-\lambda(t+x))/2} x N | + | x M c(t)|)^2 \, dt, \tag{13}$$

where

$$xN \geqslant \sup_{|x|<1} | c(t+x) - c(t)|$$

and

$$xM = \sup_{|x|<1} | e^{(-\lambda/2)(t+x)} - e^{-(\lambda t/2)} | .$$

The constant M is obviously bounded. N is bounded for the following reasons. By Assumption A1, the derivative of the technology F' is continuous. Since the feasible l's and k's are contained within a bounded set, because of the assumptions on the l's and by (6) of Lemma 1 above, then the derivative of the technology with respect to the feasible l's and k's is also bounded. Also, by (3) and (4) of Lemma 1 above, the time derivatives $\dot{k}$ of all feasible k's are bounded. Then, by assumption (vi) of the definition of C_{k_0} and equality (1) of Lemma 1, the constant N is bounded.

Since by Lemma 1 there is a uniform $\| \cdot \|_\infty$ bound C for all c's in C_{k_0}, as x goes to zero the expression in (13) and, hence that of (12), go to zero uniformly for c in C_{k_0}. Thus, condition (10) is also satisfied here. This completes the proof.

Lemmas 1 and 2 show $\| \cdot \|_\lambda$-compactness of the feasible set of consumption paths C_{k_0}. We now turn to the continuity of the welfare function:

Let $u(s, t)$ satisfy the Caratheodory condition of A3 above for $s \in R^n$ and $t \in [0, \infty)$ and let $W(s)$ be defined by

$$W(s) = \int_0^\infty e^{-\lambda t} u(s(t), t)\, dt, \qquad 1 > \lambda > 0;$$

then

Proposition 1. *W defines a continuous function from H_λ^0 to R, if and only if $| u(s, t)| \leqslant a(t) + b \mid s \mid^2$ where $a(t) \geqslant 0$, b is a positive constant and*

$$\int_0^\infty a(t)\, e^{-\lambda t}\, dt < \infty.$$

Proof. Note that $s^\alpha \to_{\|\cdot\|_\lambda} s$ if and only if $e^{-(\lambda t/2)} s^\alpha(t) \to e^{-(\lambda/2)t} s(t)$ in L_2. Let $d(t) = e^{-(\lambda/2)t} s(t)$. Then the map $s \to W(s)$ is $\| \cdot \|_\lambda$ continuous iff the map M given by

$$d \xrightarrow{M} e^{-\lambda t} u(e^{(\lambda/2)t} \cdot d, t)$$

is continuous from L_2 to L_1. By [8, Theorems 2.1 and 2.3, pp. 23–28, and remarks, p. 28] a necessary and sufficient condition for M to be continuous from L_2 to L_1 is that

$$| e^{-\lambda t} u(e^{(\lambda/2)t} d, t)| \leqslant g(t) + \alpha \mid d \mid^2$$

for $g(t) \in L_1$, which is equivalent to $| u(s, t)| \leqslant a(t) + b \mid s \mid^2$, where $a(t) = e^{\lambda t} g(t)$.

Remark. Let $0 \leqslant \zeta \leqslant \lambda$, then if $s^\alpha \to^{\|\cdot\|_\zeta} s \Rightarrow s^\alpha \to^{\|\ \|_\lambda} s$. Also, $H_\lambda^0 \supset H_\zeta^0$. Therefore if $W: H_\lambda^0 \to R$ is $\|\cdot\|_\lambda$ continuous, then $W \mid H_\zeta^0: H_\zeta^0 \to R$ is $\|\cdot\|_\zeta$ continuous. Thus, for all $0 \leqslant \zeta \leqslant \lambda$, and positive valued u, the function

$$W(s) = \int_0^\infty e^{-\lambda t} u(s(t), t)\, dt$$

is $\|\cdot\|_\zeta$ continuous, or, equivalently, for all $\zeta \geqslant \lambda$, the function

$$W(s) = \int_0^\infty e^{-\zeta t} u(s(t), t)\, dt$$

is H_λ^0-continuous.

In view of Proposition 1 we now assume $\lambda = \delta$, where δ is the discount factor of the definition of the social welfare function W.

THEOREM 1. *Under assumptions* A1, A2 *and* A3, *there exists a solution* c^* *to problem* (P) *in the set of feasible paths* C_{k_0}.

If u *is strictly concave, and* F *concave,* c^* *is also unique.*

Proof. Existence follows from Lemmas 1 and 2 and Proposition 1. Uniqueness can be established in a straightforward way using concavity assumptions on u and on the technology (A1), since C_{k_0} is a convex set if F is concave:

Let c^1 and c^2 be in C_{k_0}. Consider

$$\tilde{c} = \lambda c^1 + (1 - \lambda) c^2, \qquad 0 \leqslant \lambda \leqslant 1.$$

Then $\tilde{c} \geqslant 0$. Define $\tilde{m}$ as $\lambda m^1 + (1 - \lambda) m^2$ and $\tilde{l}$ as $\lambda l^1 + (1 - \lambda) l^2$, where m^1, m^2, l^1 and l^2 are the paths associated to c^1 and c^2 in problem (P), which exist by definition of C_{k_0}. Similarly, define k^1 and k^2 the corresponding paths in G_{k_0}, and $\tilde{k}$. Then $\tilde{m}$ and $\tilde{l}$ satisfy (b) of problem (P). Since F is concave,

$$F(\tilde{l}, \tilde{m}) \geqslant \lambda F(l^1, m^1) + (1 - \lambda) F(l^2, m^2).$$

Thus

$$F(\tilde{l}, \tilde{m}) \geqslant \tilde{c} + \beta \tilde{k} + \dot{\tilde{k}}.$$

Define

$$d = F(\tilde{l}, \tilde{m}) - \beta \tilde{k} - \dot{\tilde{k}}.$$

Then $d \geqslant \tilde{c} \geqslant 0$. Also, by definition of $\tilde{c}$ and $\tilde{k}$,

$$\beta \tilde{k} + \dot{\tilde{k}} = \lambda F(l^1, m^1) + (1 - \lambda) F(l^2, m^2) - \tilde{c} \geqslant 0$$

so that since $d \geqslant \tilde{c} \geqslant 0$ and by the above, $F(\tilde{l}, \tilde{m}) \geqslant d \geqslant 0$. Therefore, d is in C_{k_0} by definition and hence $\tilde{c}$ is in C_{k_0} also. Thus C_{k_0} is a convex set.

Also, if two paths c^1 and c^2 differ on a set of positive measure, since u is strictly concave, they cannot both be optimal, since $W(\bar{c}) > \min[W(c_1), W(c_2)]$. This completes the proof.

3. Characterization of Solutions by Prices

Definitions. $H_\lambda^{0+}[0, \infty)$ is the cone of positive paths in $H_\lambda^0[0, \infty)$, i.e. $c \in H_\lambda^{0+}[0, \infty)$ (also denoted $c \geqslant 0$) when $c(t) \geqslant 0$ a.e. for $t \in [0, \infty)$. $c > 0$ when $c \geqslant 0$ and $c(t) > 0$ a.e. on some set of positive measure, and $c \gg 0$ when $c(t) > 0$ a.e. in $[0, \infty)$. A function $f: H_\lambda^0 \to R$ is *increasing* when $f(c) > f(c_1)$ if $c - c_1 > 0$. Since by Assumption A3 u is strictly increasing, $W: (H_\lambda^0)^n \to R$ is an increasing function also.

We next prove existence of strictly positive competitive prices for c^*. First we give a definition of prices in this model.

A *price* p is an element of the dual space of $(H_\lambda^0)^{n-}$, i.e., a continuous linear real valued functional defined on $(H_\lambda^0)^n$ which is positive on positive elements of $(H_\lambda^0)^n$; $p(c) \in R^+$ is called the *value* of c in price system p. Since H_λ^0 is a Hilbert space, by definition the space of prices has the following properties:

(i) A nonzero price p must be nonzero at some period of time, i.e., $p \neq 0$ and $p \geqslant 0 \Rightarrow p(t) > 0$ on some set of positive measure of R^+, and

(ii) a price p well defines a (finite, nonnegative) value for any path of commodities c in the space, and the value of c is given by the inner product

$$\int_0^\infty e^{-\lambda t} p(t) \cdot c(t)\, dt.$$[13]

We can now prove:

Theorem 2. *Under the assumptions of Theorem 1, when u and F are concave, c^* is an optimal path in C_{k_0} with respect to W if and only if there exists a price system p^* such that*

(i) *p^* well defines a present value for all positive consumption paths c in $(H_\lambda^0)^{n+}$ given by*

$$p^*(c) = \int_0^\infty e^{-\lambda t} p^*(t) \cdot c(t)\, dt,$$

(ii) $\| p^* \| = 1$, $p^* \gg 0$, *and*

(iii) *c^* is competitive in price system p^*, i.e., c^* maximizes the value of $p^*(c)$, for all c in C_{k_0}.*

[13] Both these properties are, of course, not necessarily true, for instance, for positive prices in L_∞^*.

Proof. Let A_{c^*} be the set of all paths c in $(H_\lambda{}^0)^n$ with

$$W(c) > W(c^*).$$

By the assumptions on W, A_{c^*} is a convex subset of $(H_\lambda{}^0)^n$. Since W is continuous on $H_\lambda{}^0$ by Proposition 1, A_{c^*} has a nonempty interior. Since u is strictly increasing by A3, if $d > c^*$, $d \in A_{c^*}$ i.e., the cone $\{(H_\lambda{}^0)^{n+} + c^*\} \subset A_{c^*}$. Also since c^* is optimal in C_{k_0}, $C_{k_0} \cap A_{c^*} = \varnothing$. By [5, Theorem 12, p. 412], there exists a nonzero linear functional p^* which separates C_{k_0} and A_{c^*}. Since $\{(H_\lambda{}^0)^{n+} + c^*\} \subset A_{c^*}$, p^* can be taken to be a positive linear functional. By [12, Theorem 5.5, p. 228], this implies that p^* is continuous. Suppose now that there exists an open set $U \subset [0, \infty)$ such that $p^*/U = 0$. Since $p^* > 0$, then for some open set $V \subset [0, \infty)$, $p^*/V > 0$; define

$$z = c^* - \epsilon\phi_V + \phi_U,$$

where ϕ_V and ϕ_U denote the characteristic functions of the sets U and V, respectively, in $(H_\lambda{}^0)^n$. For ϵ sufficiently small $z \in A_{c^*}$, but

$$p^* \cdot z = p^* \cdot c^* - \epsilon p^* \cdot \phi_V < p^* \cdot c^*,$$

which is a contradiction. Hence, $p^* \gg 0$. Note that the same reasoning is valid for $p^*/\| p^* \|_\lambda$. Sufficientcy follows from the convexity of the set C_{k_0} and A3.

DEFINITION. A feasible consumption path c in C_{k_0} is called *efficient*[14] if there exists no c_1 in C_{k_0} with $c_1 > c$.

From Theorem 2 one obtains immediately

COROLLARY 1. *Assume that $\bar{c}$ is an efficient path in C_{k_0}, which is also optimal with respect to a welfare function*

$$W = \int_0^\infty e^{-\lambda t} u(c(t), t)\, dt,$$

where u satisfies the Caratheodory conditions (see A3) and is concave (not necessarily increasing) and $W(c)$ assumes at least one value strictly larger than $W(c^*)$. Then there exists a price system $\bar{p}$ such that:

(i) *$\bar{p}$ well defines a present value for all positive consumption paths c in $(H_\lambda{}^0)^{n+}$ given by*

$$\bar{p}(c) = \int_0^\infty e^{-\lambda t} \bar{p}(t) \cdot c(t)\, dt,$$

[14] Note that an optimal path may not be efficient, unless the welfare criterion W is increasing, i.e., $c' > c \Rightarrow W(c') > W(c)$. Also note that for a given welfare function W, an efficient path may not be optimal with respect to W.

(ii) $\|\bar{p}\| = 1$ *and* $\|\bar{p}\| > 0$.

(iii) If $c_1 > \bar{c}$, $\bar{p}(c_1) > \bar{p}(\bar{c})$, *and*

(iv) $\bar{c}$ *maximizes the value of* $\bar{p}(c)$ *in* C_{k_0}.

Proof. This corollary is an immediate consequence of Theorem 2. In view of the topological structure of the space H_λ^0, we only need to point out that, in this case, unlike the case in Theorem 1, $\bar{p}$ is given by a nonzero linear functional which separates C_{k_0} and the minimum convex set $B_{\bar{c}}$ in H_λ^0 which contains both A_{c^*} and the set $\{H_\lambda^{0^{n+}} + \bar{c}\}$ (as defined in Theorem 2). Note that since $B_{\bar{c}}$ contains A_{c^*}, it has a nonempty interior.

Remark. Note that because of the Hilbert space structure of the space of consumption paths, if $\bar{c}$ is an efficient path in C_{k_0}, which maximizes in C_{k_0} the value of a price system $\bar{p}$ (as defined above), then this price is given by a positive function $\bar{p}(t)$ with

$$\bar{p}(c) = \int_0^\infty e^{-\lambda t}\bar{p}(t) \cdot c(t)\, dt$$

for all consumption paths $c(t)$.

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