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PATTERNS OF POWER: BARGAINING AND INCENTIVES IN TWO-PERSON GAMES*

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We introduce the concept of a strategic dictator and use it to analyse patterns of power in two-person games that arise naturally in bargaining, arbitration, and incentive problems. A strategic dictator is an agent who has the power to ensure that a Nash equilibrium outcome is his or her preferred outcome, but who may have to lie in order to do this. We discuss applications of our analysis to Stackelberg and Cournot Duopolists, to bargaining situations, and to the existence of appropriate incentive systems.

1. Introduction and examples

In this paper we consider the structure of Nash equilibria in two-person games. Examples are bargaining situations which arise naturally in the study of the negotiation between colluding firms, and also social choice problems. It is shown that the simple and apparently symmetric underlying structure of these problems gives rise to a fundamental asymmetry in the powers of the two agents. An agent's power here is defined in terms of ability to achieve whatever outcome he or she is aiming at. One aspect of this asymmetry is that all Nash equilibria of these games are also Stackelberg equilibria.

Problems with the structure that we are concerned with arise naturally in a number of areas and we begin by considering two examples.

Example 1. Consider a system whose state is described by a point in the plane, (s_1, s_2) in R^2. The state is controlled by two agents: agent i controls component s_i $(i=1,2)$ of the state, and the two agents act independently. The

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initial state of the system is a point $(s_1^0, s_2^0) = s^0$ in R^2, and it is proposed either to remain in the 'status quo' position, or else to move a distance of at least $\varepsilon > 0$ from this to a new state. The two agents have different preferences about alternative states of the system, and consequently will bargain about the new state to be chosen. The set of possible new states, S, is thus $D \cup \{s^0\}$, where D is R^2 minus a disk of radius ε, and s^0 is its centre (see fig. 1).

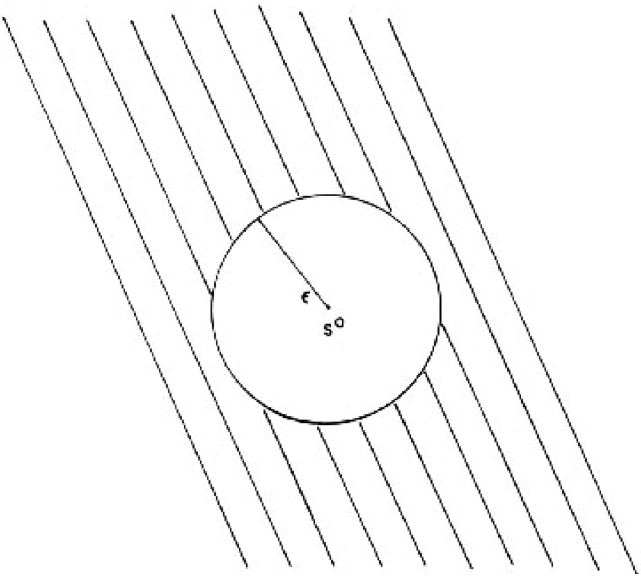

Fig. 1. Set of possible new states is the shaded area plus the 'status quo' s^0.

Each agent will choose a point s_i in S as a preferred new state: it is assumed that the agents have previously selected a *decision or arbitration rule* which, given a difference in their choices, resolves this by selecting an outcome in a predetermined way, which is felt to be fair or reasonable. Such an arbitration rule, which we assume to be continuous, is therefore a map $f: S \times S \to S$ which, given any pair of states chosen by the two agents, selects an outcome.

Example 2. Consider next a *social choice problem* when the two agents have linear, ordinal preferences on R^2. [This framework for social choice, and its extensions to non-linear preferences, are discussed in general in Chichilnisky (1982b)]. Any ordinal non-statiated preference is identified with its gradient vector, which may be normalised to be everywhere of unit length. A non-trivial linear preference is then fully identified by its unique unit gradient vector as in fig. 2. To these non-trivial linear preferences we add the 'total indifference' linear preference denoted $\{0\}$ which is, of course, satiated.

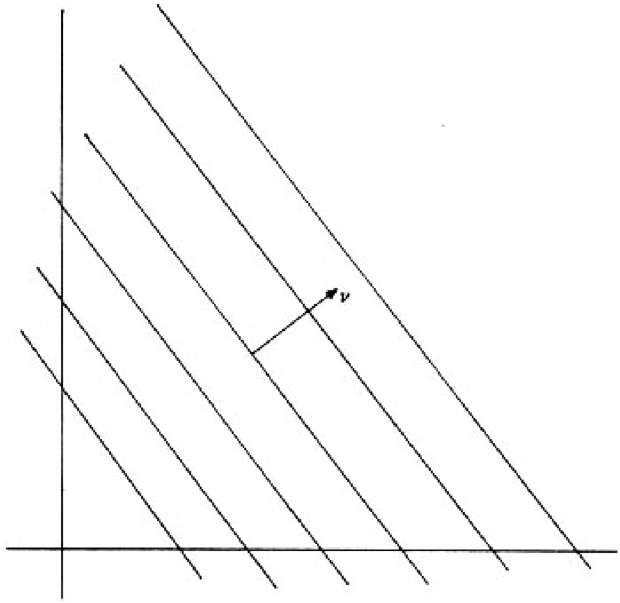

Fig. 2. A linear, ordinal non-satiated preference on R^2. v is its gradient vector.

If we admit all possible directions of increase for preferences — i.e. we work with an unrestricted domain for preferences — then the set of all possible preferences is identical with the set of all vectors of unit length, plus the indifference preference denoted 0, i.e. with the circle S^1 union $\{0\}$. Thus a social choice rule, which associates with any pair of individual preferences a social preference, is a function f:

$$f: S^1 \cup \{0\} \times S^1 \cup \{0\} \to S^1 \cup \{0\}.$$

We restrict our attention to continuous social choice rules.

The first of these two examples shows the application of our framework to a range of *bargaining problems*. These could be spatial or locational problems, where two parties are bargaining about the location, or movement, of a joint enterprise. Models of locational competition in which the state space is naturally a circle have been discussed by Heal (1980) and Salop (1979). Alternatively, one could consider models relating to *imperfect competition*. In this case, s_1 and s_2 might be the output levels of two imperfectly competitive firms. It could be supposed that the firms collude: (s_1^0, s_2^0) are the current output levels, and they are negotiating over new output levels. Clearly, the outcome eventually agreed might involve any combination of increases and decreases for the two firms, so that the space of possible new situations would be appropriately modelled as R^2 minus a disc plus the 'status quo' outcome. In this case we have a model of a *collusive duopoly*, with the function f representing the rules or institutions that they use for combining the divergent wishes of the individual firms into an outcome.

The second of the two examples mentioned above will enable us to show the applications of the present framework to problems of *strategic behaviour in social choice situations*, and to problems of *incentive compatibility of social choice rules.*

What these two examples have in common is the following structure. Each agent has a strategy space, S, which is either the unit circle in R^2 (denoted by S^1) union $\{0\}$, or is R^2 minus a small disc, union $\{0\}$. This is in fact a figure which is continuously deformable into the unit circle union its centre point and indeed is homotopic to the circle and its centre.[1] In this sense the two are topologically equivalent, and as all the arguments to be used below depend only on the general topological type of the strategy space, and not on its geometric details, we can without loss of generality take the strategy and outcome spaces to be the unit circle plus its centre $\{0\}$.

There are a number of economically very different objects that can readily be identified with such strategy spaces, and which will be considered below. One is, as mentioned, the family of all linear ordinal preferences on R^2. Another quite distinct preference family permitting the same identification is constructed as follows. Consider preferences on R^2 whose indifference curves are closed concentric circles, the centres of which will be referred to as 'bliss points'. Within this family, every preference can be identified by its bliss point, a point in R^2. Now consider the set of all preferences with bliss points either at some status quo point, or at least a distance $\varepsilon>0$ from this. Clearly this also permits the identification discussed above.

2. Notation

Agent i, $i=1,2$, chooses a strategy s_i in a strategy space S_i. Each space S_i is assumed to be homotopic to $S^1 \cup \{0\}$, the unit circle in R^2 union its centre $\{0\}$. As all arguments are topological, we shall without loss of generality suppose the strategy spaces S_1 and S_2 and the outcome space to be $S^1 \cup \{0\}$. We shall therefore be concerned with functions

$$f: S \times S \to S, \quad \text{where} \quad S = S^1 \cup \{0\},$$

which are continuous in the usual topology on R^2.

[1]Two topological spaces A and B, $B \subset A$ are said to be homotopic if there exists a family of parameterised continuous functions from A to itself, $f(\cdot, t): A \to A$, $t \in (0,1)$, such that for $t=0$, f is the identity map of A, and for $t=1$, f maps all of A onto B, i.e. $f(x,0)=x$ for all x in A, and $f(A,1)=B$.

We introduce two concepts which describe the power of the agents. The first is standard. Let S be the strategy space defined above. We say that an agent — without loss of generality agent 1 — is a *dictator* if:

$$f(s_1, s_2) = s_1, \quad \text{for all} \quad s_2 \text{ in } S.$$

A weaker concept, which is used to study *patterns of power* in two-person games, is that of a *strategic dictator*. An agent, again w.l.g. agent 1, is a strategic dictator if:

> for any s^* in S^1, and any strategy s_2 in S^1, there exists a response $s(s^*, s_2)$ in S^1, such that the outcome $f(s, s_2)$ is s^*.

This says that given a target or desired outcome s^* in S^1, then whatever strategy s_2 agent 2 chooses, there exists a strategy choice for player 1 which ensures s^* as the outcome. In general this choice depends upon s^* and upon 2's choice. If it does not depend upon 2's choice, and if $s = s^*$, then we have the usual concept of dictatorship.

An agent who is a strategic dictator has the power to attain any target outcome, once the move of the other player is known. Suppose that f is a game form, and that the definition of the game is completed by defining agents' preferences over outcomes. Then if an agent is a strategic dictator, the outcome associated with a Nash equilibrium of that game will be whatever this strategic dictator wishes it to be. A strategic dictator can therefore force any desired outcome as a Nash equilibrium.

In the next section we demonstrate that in games of the type discussed in the introduction there is *always* a strategic dictator. We then discuss the existence of Nash equilibria for such games.

In the appendix we define formally the concepts *of citizen's sovereignty*, *respect of unanimity*, a *Pareto condition*, and a *convexity condition*, which are used in the results of the following section, and are standard concepts from the literature.

3. Results and applications

Our concern in this section is to analyse the existence and uniqueness of a strategic dictator: this in effect characterises the distribution of power between agents. Following the main results we present several applications.

Theorem 1. Consider a continuous two-person game form (or arbitration rule) $f: S \times S \to S$, which satisfies the citizen's sovereignty condition. Then there exists a strategic dictator.

For a proof see the appendix. The following applications will illustrate this theorem. It is worth pointing out that since the citizen's sovereignty condition is implied by, but is weaker than, respect of unanimity, the theorem is also valid if the game respects unanimity.

When we require a stronger condition from the game or arbitration rule, namely that it also satisfies the convexity condition, we can refine this existence result to the existence *and* uniqueness of a strategic dictator. The following theorem is also illustrated in the applications:

Theorem 2. Consider a continuous two-person game form (or arbitration rule) $f:S\times S\to S$, which satisfies the citizen's sovereignty condition and the convexity condition. Then there is exactly one agent who is a strategic dictator.

For a proof see the appendix.

It may be useful to discuss a difference between the results of theorem 1 and of theorem 2. Since in theorem 1 we prove existence of a strategic dictator, and theorem 2 proves that under certain conditions there is a unique strategic dictator, a natural question is whether it is possible to have *two* strategic dictators. The answer to this is positive. Fig. 3 illustrates such a game. Player 1 always has a strategic answer to player 2 in order to attain

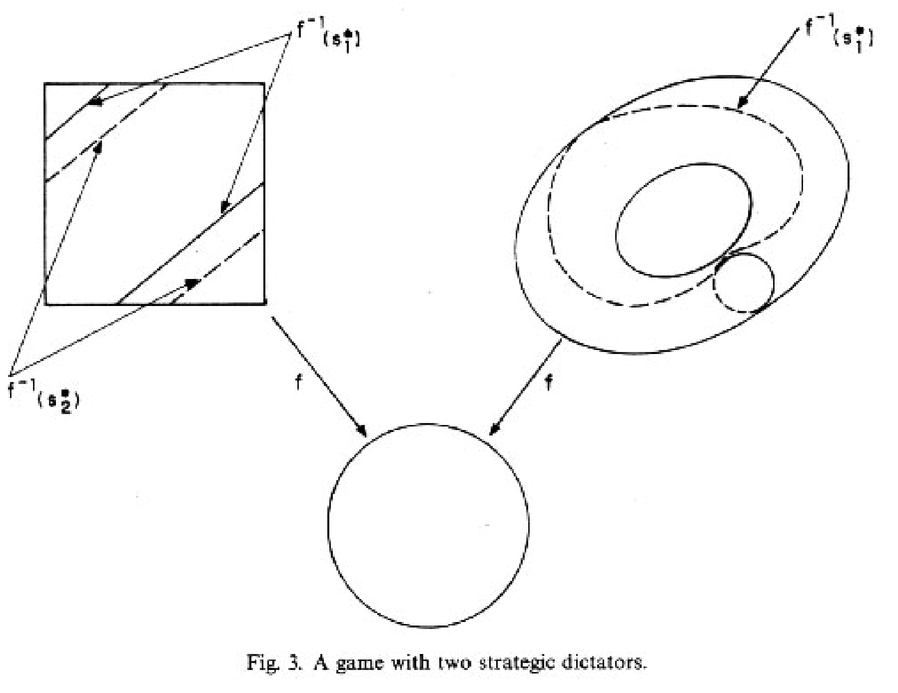

Fig. 3. A game with two strategic dictators.

his/her preferred outcome, and player 2 is also in the same situation. The pattern of power in such a game is therefore rather symmetric: the outcome depends only on who makes the first, and who makes the second, move. In this case the difference in outcomes lies *not* in the power of the agents, but rather in the information available to each player. However, since the concept of Nash equilibrium concerns a symmetric informational structure, it becomes clear that we have a problem with existence of such an equilibrium when the two players are strategic dictators. We study this formally below.

The game in fig. 3 is given by the function $f: S^1 \times S^1 \to S^1$ and $f(0,0)=0$. Fig. 3 illustrates the hypersurfaces of the map f. The strategy space of each player is S^1. Therefore the product of the strategy space is $S^1 \times S^1$ as in the right-hand side of fig. 3. The left-hand side is a flat map of the space $S^1 \times S^1$. $f^{-1}(s_1^*)$ denotes the reaction function of player 1 with preferred outcome s_1^*; $f^{-1}(s_2^*)$ represents the same object for player 2. Note that both players are strategic dictators. For instance, for any strategy s_2 stated by player 2, player 1 can find a response $s(s_2^*)$ such that $f(s(s_2), s_2) = s_1^*$, where s_1^* is the preferred outcome of the first player.

In order to study the existence of Nash equilibria, we first characterise agents' reaction functions. Consider first an agent (say agent 1) who is a strategic dictator. s_1^* is the outcome which agent 1 ranks highest, called 1's bliss point, and we know, by definition, that given any strategy s_2 for agent 2, there exists an s_1 for 1 such that $f(s_1, s_2) = s_1^*$. Clearly, s_1 depends upon s_2, $s_1(s_2)$, and this function $s_1(s_2)$ is in fact agent 1's reaction function. It gives 1's best response to any move by 2. But as $f(s_1(s_2), s_2) = s_1^*$, it is clear that the graph of 1's reaction function (or correspondence) must lie in the pre-image of s_1^* under f, denoted $f^{-1}(s_1^*)$, i.e.

$$\{s_1(s_2), s_2\} = f^{-1}(s_1^*).$$

The graph of the reaction correspondence of a strategic dictator is therefore the inverse image of that agent's highest ranked outcome. This fact leads immediately to the following elementary remark about existence of Nash equilibria.

Lemma 3. Let f be a game in which both agents are strategic dictators, with preferred actions s_1^* and s_2^*, and $s_1^* \neq s_2^*$. Then there is no Nash equilibrium.

This is immediate. Note that the Nash equilibrium is contained in the intersection of the two reaction correspondences. As these are respectively $f^{-1}(s_1^*)$ and $f^{-1}(s_2^*)$, and f is a function, this intersection is always empty when $s_1^* \neq s_2^*$.

It would be nice to be able to prove the converse of the above remark, namely that if there is only one strategic dictator, then a Nash equilibrium does exist. Unfortunately this is not true: there are situations where there is only one strategic dictator, and yet there is no Nash equilibrium. In order to ensure the existence of an equilibrium, restrictions must be placed on the preferred outcomes s_1^* and s_2^* which in a sense require that preferences should not be directly opposed.

In preparation for our first application we state the following result, in which preferences are distance functions from bliss points, while the space of strategies is still left unchanged $(=S)$, as discussed in section 1.

Theorem 4. *Let $f:S\times S\to S$ be a continuous game form satisfying the citizen's sovereignty condition and let preferences over outcomes be given by families of concentric circles centred on agents' bliss points. Then if the game thus defined has a Nash equilibrium, the outcome associated with it is always identical to the bliss point of one of the agents.*

We can summarize here how this result is obtained. By theorem 1 at least one agent is a strategic dictator. However, from theorem 3 it follows that if there is a Nash equilibrium, there is only one strategic dictator. Hence, we have one strategic dictator and, by definition, the Nash equilibrium outcome will be this agent's bliss point.

It is worth emphasising that theorems 1 and 4 do not require that preferences be linear, or given by distance functions. The crucial feature is, rather, that the space of strategies be (topologically equivalent to) $S=S^1\cup\{0\}$. Clearly, both linear and circular-distance preferences can be identified with elements in S: in the case of linear preferences, each preference is identified with its (unique) gradient in S. In the case of preferences given by concentric circles, each preference is identified with its bliss point, an element in S as well.

3.1. Application 1: Stackelberg and Cournot solutions

We can now discuss the applications of theorems 1 to 4 above. Theorem 4 emphasises that within the present framework, a Nash equilibrium will be characterised by a very asymmetric distribution of gains at the outcome. It is probably true that a Nash equilibrium is not normally thought of in this way: it is thought of as a symmetric equilibrium concept, in contrast with say a Stackelberg equilibrium, where there is a clear difference between the positions of the leader and the follower. In fact, in the present context these two concepts can be shown to be the same. If the strategic dictator was a leader in a Stackelberg equilibrium, and was choosing the best point in the other agents reaction function, then clearly he or she could do no better than to choose a point in the pre-image of the preferred outcome. This is precisely the outcome that would result from a Nash equilibrium. Conversely, if the agent (say agent 1) who was not a strategic dictator, was the leader in a Stackelberg game, and was choosing the best point on 2's reaction function, then as this is contained in $f^{-1}(s_2^*)$, agent 1 could do no better than to pick a point yielding the preferrred outcome of the strategic dictator. We can therefore establish:

Corollary 5. Consider the game of theorem 4. Then all Nash equilibria are also Stackelberg equilibria.

Proof. Theorem 4 and the argument above demonstrates that the set of Nash equilibria is contained in the set of Stackelberg equilibria, as required.

The next applications can be considered under three headings — bargaining and arbitration, implementation of social choice rules, and strategic misrepresentation of preferences.

3.2. Application 2: Bargaining

Consider first an arbitration or bargaining problem such as that set out in example 1 of section 1. A system whose state is described by a point $s=(s_1,s_2)$ in R^2 is initially at a position $s^0=(s_1^0,s_2^0)$. The two agents controlling the system each have preferences over the set S of alternative positions to which it might move, and S is homeomorphic to $S^1 \cup \{0\}$, the unit circle in R^2 union its origin. s_1 and s_2 are positions that the agents announce as their preferred new positions, and the outcome is then given by $f(s_1,s_2)$, where f is a continuous function representing the arbitration or decision rule adopted. It is clearly reasonable to suppose that f respects unanimity, so that if both agents agree on the new position, this agreed alternative is selected. Most axiomatizations of the arbitration or bargaining process will certainly satisfy this condition.

Consider now a bargaining or arbitration situation where each agent knows the rule f and plays strategically, i.e. announces a desired outcome which may not be the true bliss point but is chosen so as to yield the best possible outcome for that agent. Then from theorems 1 and 2 we know that if f satisfies citizen's sovereignty and the convexity condition, again a reasonable requirement of arbitration rules, there is a fundamental asymmetry of power between the two agents, one and only one being a strategic dictator. Furthermore, we know from theorem 4 that if this bargaining game has a Nash equilibrium, then it will be one yielding as an outcome the strategic dictator's bliss point. The system will in this case operate as if this agent were a dictator, in the sense that if he or she were to change bliss point, the outcome would change identically.

3.3. Application 3: Incentives

We turn next to the application of the results of the previous section to the problem of incentive compatability, in the implementation of social choice rules. We work here with the framework of example 2 of section 1: individuals have ordinal linear preferences on R^2, so that the space of individual preferences is S, the circle in R^2 union its centre. A social choice rule is then a continuous map $f:S \times S \to S$ which associates a social preference with any pair of individual preferences.

Suppose that individuals have message or strategy spaces M and that a game form is a continuous function $g:M \times M \to S$. Individuals do not know the social choice rule f which a central agency wishes to implement, but do know the game form g. Each individual also knows the strategies chosen by others, though not their true preferences. Then we say that g implements the social choice rule f in Nash equilibria if the diagram (fig. 4) commutes. Here Eg is the Nash equilibrium correspondence of the game defined by g and the agents' rankings over outcomes in S. These rankings are assumed to have a very simple form: an outcome a is ranked by agent i above an outcome b if and only if a is nearer than b to the agent's true bliss point, using the usual Euclidean distance measure. Eg thus assigns to any pair of individual preferences in $S \times S$ a set of messages or strategy pairs in $M \times M$ which form Nash equilibria, given the game form g.

We shall assume the strategy or message space M to be homeomorphic to (i.e. of the same topological type as) the preference space, and so without loss of generality identify M with S. A particular case of this is when strategies or messages are announcements which purport to describe the agents' true preferences. Hence, g: $M \times M = S \times S \to S$. We shall also suppose g to satisfy

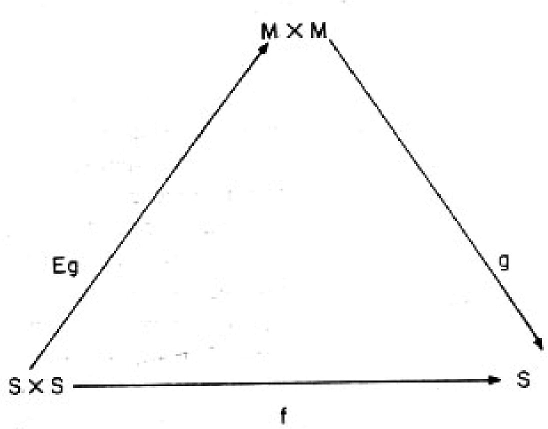

Fig. 4.

citizen's sovereignty (so that the image of g restricted to the diagonal of $S \times S$ covers the outcome space), which means that the agents by coordinating their strategies can ensure any possible outcome. Within this framework, we can now establish the following:

Corollary 6. A social choice rule f is implemented in Nash equilibria by a game g satisfying the citizen's sovereignty condition if and only if f is dictatorial.

Proof. Suppose fig. 4 commutes. Then the game given by g and individual rankings over outcomes has, by assumption, a Nash equilibrium. Hence, by theorem 4, the Nash equilibrium outcome is the preference of one of the agents. This implies immediately that f is dictatorial. The sufficiency part of the argument is obvious.

Corollary 7. A social choice rule can be implemented via cooperative strong equilibria[2] only if it is dictatorial.

This follows directly from theorem 6 since the set of strong equilibria of a game is contained in the set of Nash equilibria.

3.4. *Application 4: Manipulation of social choice rules*

Our final application is to the manipulation of social choice rules. The issue in this case is whether, if individuals know the social choice rule f, truthful revelation of their preferences will be the best strategy for them. If the social choice rule satisfies citizen's sovereignty, then theorem 1 can be used to answer this question immediately. It assures us that at least one agent is a strategic dictator. If such an agent is not actually a dictator, then it is clear that he or she will in general be able by misrepresentation of preferences to obtain an outcome preferable to that which would result from truthful revelation.

4. Relationship with other results

Our final task is to trace out the connections between the results of the previous two sections, and results already available in the literature. The basic results are theorems 1–4 which are about the characterisation and existence of Nash equilibria and of strategic dictators. There is no clear precedent for these, although they represent a natural development of

[2] *A strong equilibrium* is a set of strategies which cannot be upset by any coalition. A coalition M upsets a vector s of strategies if there exists a feasible vector s' of strategies with $s'_i = s_i$ for $i \notin M$, and s'_i preferred or indifferent to s_i for all i in M and strictly preferred for some i in M.

Chichilnisky's (1979) concept of 'topological equivalence to dictatorship'. These results exploit the fact that under certain circumstances it is natural for individual strategy sets in a game to be topologically non-trivial (i.e. non-contractible, and so in particular non-convex). This gives a very distinctive mathematical structure to the resulting game, and there are of course few if any results available on the existence and characterisation of Nash equilibria of games with non-convex strategy sets. It is shown that this mathematical structure gives rise naturally to the concept of a strategic dictator, an agent who in many respects has as much power as a normal dictator, but who has to exercise a little ingenuity in order to exploit this power.

Those parts of our results which do have a clear relationship with existing literature are those that deal with the implementation and manipulation of social choice rules. Here we work with continuous social choice rules satisfying the citizen's sovereignty condition and defined on unrestricted domains of preferences, and by applying the concept of strategic dictator we establish essentially two main results.

The first is that a rule is Nash implementable if and only if it is dictatorial. The second is that a rule is manipulable (in the sense that truthful revelation is not a dominant strategy) unless it is dictatorial. The first of these results is clearly related to those of Roberts (1977) and Dasgupta, Hammond and Maskin (1979) on the impossibility of Nash implementation with unrestricted domains, though the frameworks are different and we deal here only with the two person case. The result on manipulation is clearly connected to those of Gibbard (1973) and Satterthwaite (1975), though again there are differences in the frameworks which make detailed comparisons difficult.

We should perhaps end with a comment on the restriction of our results to the two-person case and two-dimensional state space. The techniques used here — the degree of a map from S^1 to S^1 — are sufficient to prove the results in these cases. These results could also be extended to more general frameworks, but this would require the use of more powerful topological techniques. An extension of the results to this case might therefore have a less favourable ratio of effort to insight. It is also true that the two-dimensional case appears to have considerable economic interest, both in view of the fact that locational problems are naturally two-dimensional, and in view of the fact that in the study of games and bargaining situations those involving two agents have always been felt to merit particular attention.

Appendix

We shall say that a function f *respects unanimity* if:

$$f(s,s)=s, \quad \text{for all } s \in S.$$

The *degree* of a continuous map $f:S^1 \to S^1$ is the net number of times the image of S^1 under f 'wraps completely around' S^1. Thus, if x is a position on S^1 measured in radians, the map $f(x)=x$ has degree one, and $f(x)=nx$ has degree n. Note that:

$$\Delta = \{(s_1, s_2) \in S^1 \times S^1 : s_1 = s_2\},$$

i.e. the 'diagonal' of $S^1 \times S^1$ is homeomorphic to the circle S^1. Hence, if

$$f: S^1 \cup \{0\} \times S^1 \cup \{0\} \to S^1 \cup \{0\},$$

we can talk of the degree of f when restricted to Δ, degree f/Δ. We shall say that a function f satisfies *citizen's sovereignty* if degree $f/\Delta = 1$. Note that citizen's sovereignty is implied by the conditions of respect of unanimity as this latter implies that $f/\Delta =$ identity, and so in particular f has degree one. Citizen's sovereignty is of course *weaker* than respect of unanimity, as it implies only that f restricted to the diagonal covers S^1, and not that it is the identity. We call this condition 'citizen's sovereignty' because it implies that by coordinating their strategies, the players can achieve all outcomes.

In the case of applications of our framework to social choice theory, we shall be interested in the *Pareto condition*. Let $s_1 \in S$ and $s_2 \in S$ be linear preferences on R^2. Recall $S = S^1 \cup \{0\}$. Then a social choice rule $f: S \times S \to S$ is said to satisfy the Pareto condition if whenever both s_1 and s_2 rank alternative x above alternative y, then the social preference $f(s_1, s_2)$ also ranks x above y. Geometrically, this means that $f(s_1, s_2)$, and also $f(s_2, s_1)$, must be contained in the cone of vectors that have non-negative inner products with all vectors having non-negative inner products with both s_1 and s_2. This is the cone $C(s_1, s_2)$ determined by s_1 and s_2 and shown in fig. 5. Thus, f is Pareto if and only if $f(s_1, s_2)$ and $f(s_2, s_1)$ are both in $C(s_1, s_2)$ for all $(s_1, s_2) \in S^1 \times S^1$.

Consider next the function:

$$f(\cdot, s_2^0): S^1 \to S^1,$$

which for a fixed value of s_2 sends S^1 to S^1 as s_1 varies. Then if f satisfies the Pareto condition, the degree of $f(\cdot, s_2^0)$ is at most one for any s_2^0. This is because in the case where s_2 is fixed, as s_1 assumes all values in S^1, the image can wrap around S^1 completely at most once.

For any s_1 and s_2 in S^1, we shall define conv.(s_1, s_2) as the *circular convex hull* of s_1 and s_2. This is the smallest interval in S^1 containing s_1 and s_2. Formally, if $(s_1, s_2)^+$ is the anti-clockwise interval containing s_1 and s_2, and $(s_1, s_2)^-$ the clockwise interval, conv.(s_1, s_2) is the smaller of $(s_1, s_2)^+$ and $(s_1, s_2)^-$. If both are equal, conv.(s_1, s_2) is the whole circle S^1. We now

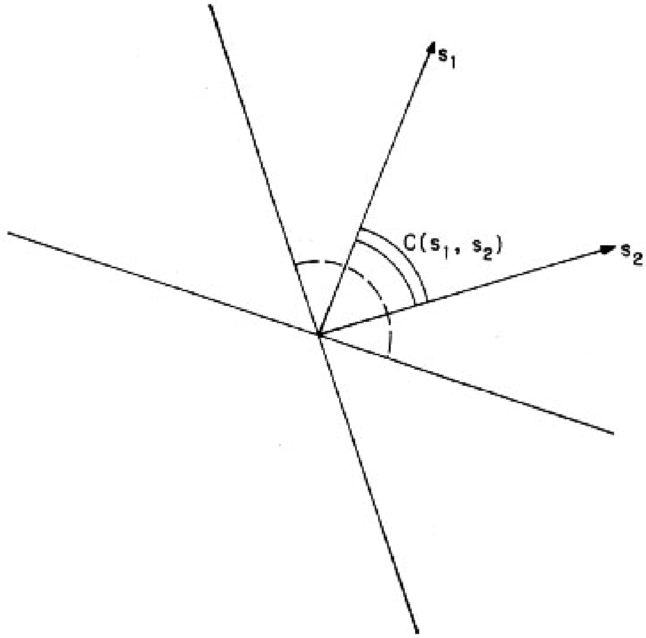

Fig. 5. s_1 and s_2 are preferences. All vectors having positive inner product with both, lie in the cone $C(s_1, s_2)$, which is shown by the solid lines.

introduce the *convexity condition* on f, which is that for all s_1 and s_2, $f(s_1, s_2) \in \text{conv.}(s_1, s_2)$. The reader can readily verify that if f is a social choice rule mapping individual preferences into a social preference, the convexity condition and the Pareto condition are equivalent. It is also immediate that the convexity condition, like the Pareto condition, implies that for any s_2^0, the degree of $f(\cdot, s_2^0)$ is at most one.

A topological space X is *connected* when it does not contain two subsets $A \subset X$, $B \subset X$, such that $A \cup B = X$, $A \cap B = \varnothing$, and both sets A and B are open and closed simultaneously. A *connected component* of X is a connected subset of X which is not contained strictly in any other connected subset of X. The space $S^1 \cup \{0\}$ has two connected components. A continuous map $f: X \to Y$ must map a connected component of X into a connected component of Y.

Proof of theorem 1. We have a function:

$$f: S^1 \cup \{0\} \times S^1 \cup \{0\} \to S^1 \cup \{0\}.$$

Since f satisfies the citizen's sovereignty condition, it follows that the image of the restriction of f on $S^1 \times S^1$ must cover S^1 (for, otherwise, its degree would be zero on Δ). For instance, in the case when f respects unanimity, the image of the restriction of f on $S^1 \times S^1$ covers S^1, since in this case $f(x,x)=x$, $\forall x$ in S^1. Now, by continuity of f, the image of the set $S^1 \times S^1$ which is a connected component of $S^1 \cup \{0\} \times S^1 \cup \{0\}$, must be contained in one connected component of $S^1 \cup \{0\}$, as defined above. Therefore, since $f(S^1 \times S^1)$ must cover S^1, it follows that $f(S^1 \times S^1) \subset S^1$. Therefore, in the following we restrict ourselves to studying the map restricted to $S^1 \times S^1$, i.e.

$$f: S^1 \times S^1 \to S^1.$$

The product space $S^1 \times S^1$ is also called the two-dimensional *torus*, and it can be represented by the unit square with the opposite sides identified, as in fig. 6.

Let Δ denote the diagonal of $S^1 \times S^1$, $\Delta = \{(s_1, s_2) \in S^1 \times S^1 \colon s_1 = s_2\}$. For a given $s_0 \in S^1$ let A denote the set $\{(s_0, s)\colon s \in S^1\}$, and B the set $\{(s, s_0)\colon s \in S^1\}$.

By the condition of citizen's sovereignty:

$$\text{degree } f/\Delta = 1. \tag{A.1}$$

Now let T denote the 'triangle' whose boundary is $\Delta \cup A \cup B$, as in fig. 1. Since f is defined continuously over all T, then the degree on the boundary

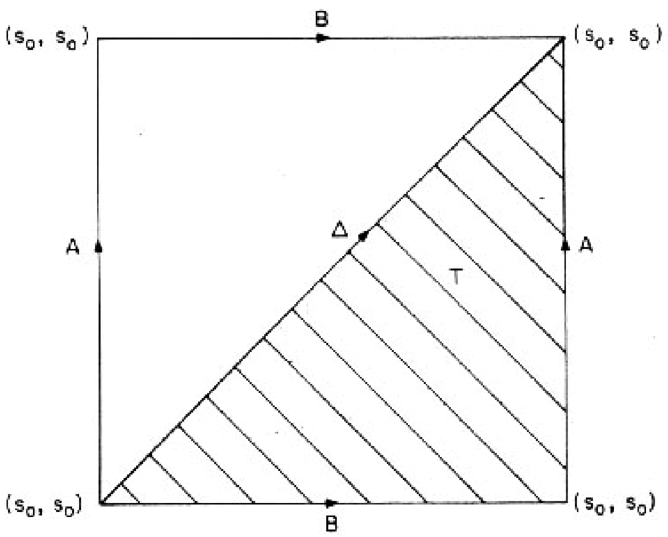

Fig. 6. The space of strategy profiles with two voters and two-dimensional choice space: $S^1 \times S^1$.

of T must be zero, i.e.

$$f/\Delta \cup A \cup B = 0; \tag{A.2}$$

proof of this statement can be found in Chichilnisky (1979) where it is also shown to be related to Brouwer's fixed point theorem. Since Δ, A, and B are all circles with one point in common, it follows that

$$\text{degree } f/\Delta \cup A \cup B = \text{degree } f/\Delta + \text{degree } f/A + \text{degree } f/B. \tag{A.3}$$

Since by (A.1) degree $f/\Delta = 1$, (A.3) implies that either degree f/A or degree f/B must be different from zero. Without loss of generality, assume that degree $f/A \neq 0$. This implies that for any fixed $s_0 \in S^1$ and any s_2^* in S^1, there exists some $\hat{s}_2 \in S^1$ such that $f(s_0, \hat{s}_2) = s_2^*$. By continuity of f for all $s_0' \in S^1$, if $A' = \{(s_0', s): s \in S^1\}$, then the degree of f on $A' \neq 0$ because the degree (modulo 2) of a continuous map is preserved under continuous deformations, and A' is a continuous deformation of A in $S^1 \times S^1$ [see, for example, Spanier (1966, p. 54)]. Therefore, we have shown that for any arbitrary strategy s in S^1 of the first agent, and *any* outcome $s_2^* \in S^1$ desired by the second agent, there exists an $\hat{s}_2$, such that the outcome $f(s, \hat{s}_2) = s_2^*$.

If for all s^*, $\hat{s}_2 = s_2^*$, then the second agent is a dictator. If for some s^*, $\hat{s}_2 \neq s_2^*$, then the second agent is a strategic dictator. This completes the proof.

It is important to note that there may be two strategic dictators, as in fig. 3 in the text. It is logically quite possible that given a strategy choice by agent 1, there exists a choice for 2 which will ensure any outcome 2 prefers, and simultaneously that given a choice by 2, then there exists a strategy for 1 yielding any outcome 1 prefers. In this case, if one thinks of a game where players move sequentially and the latter player knows the move made by the former, then it is clear that the second player is in a very advantageous position.

Proof of theorem 2. By theorem 1 there is at least one strategic dictator. We therefore need to show that there is at most one. We know that

$$\text{degree } f/\Delta = -\text{degree } f/A - \text{degree } f/B.$$

By the condition R, degree $f/\Delta = 1$, and by the convexity condition degree f/A is either zero or one. But in the equation

$$1 = -\text{degree } f/A - \text{degree } f/B$$

the only solutions with the degree of f on A and B either zero or one, are degree $f/A=0$, degree $f/B=1$, and vice versa. Now if f satisfies the convexity condition and degree $f/A=0$, then f restricted to A cannot cover S^1. Hence, only B can be a strategic dictator, as required.

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