Journal of Mathematical Economics 9 (1982) 207–221. North-Holland Publishing Company

STRUCTURAL INSTABILITY OF DECISIVE MAJORITY RULES

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Received September 1979, final version received May 1981

We study social aggregation rules that satisfy a Pareto and a decisive majority condition. A rule satisfies the decisive majority condition if whenever the voters can be divided into two internally homogeneous groups (i.e. groups within which all individuals agree about all possible choices) and these groups have opposite preferences, then the majority's preference is respected. No assumptions are made about the outcome when the individuals have more than two different preferences even when a majority agrees on a choice x being preferable to another y. Therefore a decisive majority condition does not imply, i.e., is strictly weaker than, majority rules. The main result is that Pareto decisive majority rules are necessarily structurally unstable, in the sense that the outcome preference, which is a vector field, will undergo major changes in structure in response to small changes in the underlying parameters. The importance of structural instability derives from the fact that small errors of observation will lead to drastically different answers. An example of the result for three voters and with a two-dimensional choice space is also given.

1. Introduction

Majority rules have an obvious appeal as voting procedures. However, it has been known for a long time that they may contradict desirable properties for aggregation of individual preferences. In particular, Condorcet's paradox (1785) has shown that majority rules are not consistent in general with the property of transitivity of preferences.

It can also be proven that majority rules are unstable in the sense that small shifts of the individual preferences may lead to a regrouping of majorities and therefore to significant changes in the outcome of the vote. This type of instability of the outcome with respect to underlying parameters is usually called structural instability. It is a corollary of our results here that majority rules are in general unstable in this sense.[1]

*This research was supported by the UNITAR Project on Technology, Distribution and North–South Relations, and carried at the Center for Social Sciences at Columbia University. I thank G. Debreu, R. Findlay, P. Hammond, H. Polemarchakis, L. Taylor, H. Varian, M. Hirsch, G. Heal and a referee for helpful suggestions.

[1]See for instance a recent article of Schofield (1978) where a condition is given that guarantees the existence of a local cycle for majority voting rules; this condition is an extension of an earlier one given by Kramer. Instability in their sense refers to the existence of cycles. We, instead, use a concept of structural stability akin to one used in mathematical physics and biology, to detect

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It could be argued, however, that majority rules are too restrictive, especially in view of the Condorcet paradox and much of the subsequent work in social choice theory.

A much less demanding requirement on the voting procedure is that it be Pareto,[2] and that whenever the voters can be divided into two internally homogeneous groups with opposite preferences, then the majority's preference be respected. We call such voting procedures decisive majority rules.

Sen (1970) has shown that a decisive voter[3] is a dictator when Arrow's axiom of independence of irrelevant alternatives is accepted. A decisive majority rule would in that case also be a "dictatorship of the majority". However, the axiom of independence of irrelevant alternatives is generally considered to be too strong, and in this paper we do not require it. It should be emphasized that a decisive majority rule need not be a majority rule in our case.[4] As a consequence, we do not require the number of voters to be odd.

We study here two axioms on the rule that aggregates individual preferences into a social preference. The rule must be Pareto, and majorities must be decisive. We prove that any aggregation rule satisfying those two axioms must necessarily be structurally unstable, namely, it must exhibit drastic changes in the outcome as the individual preference undergo small perturbations. Pareto and decisive majority axioms are therefore inconsistent with a form of continuity of the aggregation or voting procedure. This discontinuity implies, in particular, that sufficient statistics for aggregating preferences do not exist, and that 'almost equals' are treated very differently.

We use here techniques that were developed in Chichilnisky (1980, 1981) for the study of Social Choice problems with topological tools. In Chichilnisky (1980) it was shown that social rules that respect unanimity and anonymity must be unstable. The results implied the lack of contractibility of certain spaces of preferences. They are also related to the fact that no continuous global representation from certain spaces of preferences P into spaces of utility functions U can be constructed.[5]

when the structure of the problem or its solution undergo drastic changes with respect to initial parameters. In our case both the initial parameters *and* their solutions are elements of function spaces (e.g. vector fields) whose overall structure changes discontinuously with respect to the initial data, in the appropriate topology of these function (infinite-dimensional) spaces. In effect any concept of instability refers to a form of discontinuity; for instance, Liapunov instability can be interpreted in this manner given the appropriate topologies.

[2]Namely if all voters prefer outcome x to y, then so does the social rule.

[3]A voter is called decisive in Sen's framework if the outcome of the social rule agrees with that of the voter when everyone else's preference is opposed to that outcome.

[4]While majority rules, instead, clearly satisfy the decisive majority conditions.

[5]Since otherwise the representation of preferences by utilities $R:P\to U$ composed with the convex addition of utility functions could yield a social aggregation rule of preferences that respects unanimity and anonymity. The topology on smooth preferences used here is a C^1 sup

This paper studies a different problem, concentrating instead on the topological properties of the Pareto and decisive majority conditions. The Pareto condition is strictly stronger than the condition of respect of unanimity, and the condition of anonymity is of a different nature than the decisive majority condition. It should be noted that the lack of existence of global continuous representations from preferences to utilities does not in itself resolve the question of whether or not stable decisive majority rules exist. The fact that a global continuous representation of preferences does not exist, does not imply that decisive majorities are structurally unstable.

The results are obtained here by studying the topological degree of maps induced by the social aggregation rule. The Pareto and decisive majority conditions are proven to induce contradictory properties on the topological degree of these maps.

The rest of this paper is organized as follows: section 2 contains notation and definitions. In section 3 a particular example is constructed to show the instability of decisive majority rules for a 2-dimensional choice space and 3 voters. Section 4 gives a proof of the general theorem.

2. Notations and definitions

Let X be the *choice space*, such as a unit cube in R^n denoted I^n, or the positive orthant of R^n, R^{n+}. Since our framework is topological, it suffices to consider any space which is diffeomorphic to I^n.

A *preference* p on X is defined by giving for each choice x in X a preferred direction, or equivalently, the normal to the indifference surface at x, a vector denoted $p(x)$. Following a usual definition in social choice theory [e.g. Arrow (1953)] preferences are ordinal, and intensities of preferences are not considered. Therefore we normalize the vector fields that give our preferences to be of unit length, i.e., $\|p(x)\|=1$ for all x. A preference is therefore a map $x \to p(x)$ from choices x into the tangent space of X, $T(X)$, such that for each x, $p(x)$ is in the tangent space of X at the choice x. Such a map is called a

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topology which gives a desirable structure for the study of the problem, for instance makes the space of smooth preference complete, which is of course an important feature when continuity arguments are made. Any topology that defines proximity in a way that implies proximity of gradients or equivalently of 'indifference surfaces', or demand functions, would also yield the same results. A referee pointed out that for a special class of preferences, whose strict preferences have open graph *and* the sets of preferred points to a given choice is convex, the proof given in this paper would also work when the space of preferences is given the *closed convergence topology*, using a continuous selection for the correspondence mapping preferences into the set of points preferred to a given choice (which is a lower semi-continuous correspondence with this topology) instead of the map Γ defined in this paper. The existence of such a continuous selection follows from Hildenbrand (1974, corol. 3, p. 98). Note, however, that under the closed convergence topology smooth preferences do not form a complete space and therefore limits of smooth preferences may not be smooth. Therefore, when studying smooth preferences this topology does not seem desirable.

vector field on X. We topologize the spaces X and $T(X)$ as usual, and we assume that the preferences in the space P are defined by continuously differentiable vector fields; we then endow the space of continuously differentiable (bounded) vector fields $V(X)$ with the C^1 topology.[6] As in Debreu (1972) and Chichilnisky (1976) we assume that preferences are given by locally integrable vector fields, i.e., that $p(x)$ is locally the gradient of a real-valued utility function on X. The space P of all such locally integrable preferences is characterized within $V(X)$ by the Frobenius integrability conditions.[7] The Frobenius conditions and the normalization $\|p(x)\|=1$ are both closed conditions, therefore the space P endowed with the topology inherited from $V(X)$ is a complete space since it is a closed subspace of the space of vector fields $V(X)$. The space $V(X)$ is infinite-dimensional and the space P contains infinite-dimensional manifolds [see Chichilnisky (1976)].

We assume there are k voters ($k>2$). A *profile* is an ordered k-tuple of preferences of the voters, say, $(p_1,\ldots,p_k)\in P^k$, the k-fold product of P. A *social aggregation rule* is a map

$$\phi: P^k \to P,$$

that maps a profile into a social preference in P.

ϕ is said to be *structurally stable* or simply *stable* when it is a continuous map. ϕ is said to satisfy a *Pareto condition* when the following is always true: if for all voters, $1,\ldots,k$, x is preferred to y (for instance if the utility functions that represent $p_1,\ldots,p_k$ give a higher value to x than to y), then $\phi(p_1,\ldots,p_k)$ prefers x to y. The *decisive majority condition* on ϕ is that at any choice x whenever there is one homogeneous group of voters with profile

[6] We give P the C^1 topology to obtain a nice topological structure on P; in effect any topology on P that restricted to linear preferences coincides with the convergence of vectors in R^n will also give our result. See also footnote 5.

[7] The Frobenius integrability conditions are usually given by a set of partial differential equations: they are necessary (but not sufficient) conditions for a vector field to be the gradient of a real valued function. For a discussion of these conditions see, for instance, Debreu (1972).

If $g(x)$ is the normalized vector giving at each point the direction orthogonal to the indifference surface $H(x)$ of a preference at x, then g can be thought of as a function from the choice space X to the unit sphere of R^n [a continuously differentiable (C^1) function]. The integrability problem is the existence of a utility function u from X to R, such that its derivative D is everywhere a strictly positive multiple of $g(x)$, i.e.,

$$(*) \qquad Du=\lambda g(x), \quad \text{where } \lambda \text{ is a positive real-valued function on } X.$$

From (*), by equating the partial derivatives $\partial_i\partial_j(u)$ and $\partial_j\partial_i(u)$, writing similar equalities for pairs of indices (j,k) and (k,i) and eliminating λ, a necessary condition for the existence of a (C^2) utility function that satisfies (*) on $\dot{X}$ is

$$(**) \qquad \forall (i,j,k), \quad g_i(\partial_j g_k-\partial_k g_j)+g_j(\partial_k g_i-\partial_i g_k)+g_k(\partial_i g_j-\partial_j g_i)=0.$$

Condition (**) is sufficient for integrability *locally*, i.e., it implies at each point x in X the existence of a neighborhood V of x and a function u that satisfies (*) on V.

$(p_{i1},\ldots,p_{iq})$ and $p_{im}=\bar{p}$ for $m=1,\ldots,q$, and another group with profile $(p_{j1},\ldots,p_{j(k-q)})$ with $p_{jl}=-\bar{p}$ for $l=1,\ldots,k-q$, and one of the groups is a majority (e.g. $q>k-q$), then the outcome $\phi(p_{i1},\ldots,p_{iq},p_{j1},\ldots,p_{j(k-q)})$ agrees with the majority (e.g. is $\bar{p}$).

Notice that the decisive majority condition is strictly weaker than a majority condition. Decisive majorities determine the outcome only in the very particular cases in which the group of voters can be divided into two subgroups which are each completely homogeneous, i.e., within each subgroup everybody agrees with respect to all possible choices, and furthermore the two subgroups are opposed to each other. The decisive majority condition therefore puts no restriction on the outcome in cases when the voters have at least three different preferences, even when a majority of them agrees on a choice x being preferable to a choice y. A social aggregation rule is called a *decisive majority rule* when it satisfies both Pareto and decisive majority conditions.

In the following we shall use the concept of *topological degree* of a map. Before giving its formal definition, we shall describe intuitively the topological degree in the particular case of a continuous map between two circles, $f:S^1\to S^1$.

The topological degree of f denoted $\deg(f)$ can be thought of in this case as the number of times that the image of S^1 under f wraps around S^1. For example, the composition of any function f that maps the circle once onto the figure eight (on the right of fig. 1) with the map Π that projects the figure eight on the circle, has degree 2. The composition map $\Pi\circ f$ is topologically equivalent to mapping the vector v in S^1 measured in radians to the vector $2v$ in S^1. Under $\Pi\circ f$ each point in the image S^1 is covered twice. A similar map will be used in section 3 to study the case of three agents and a 2-

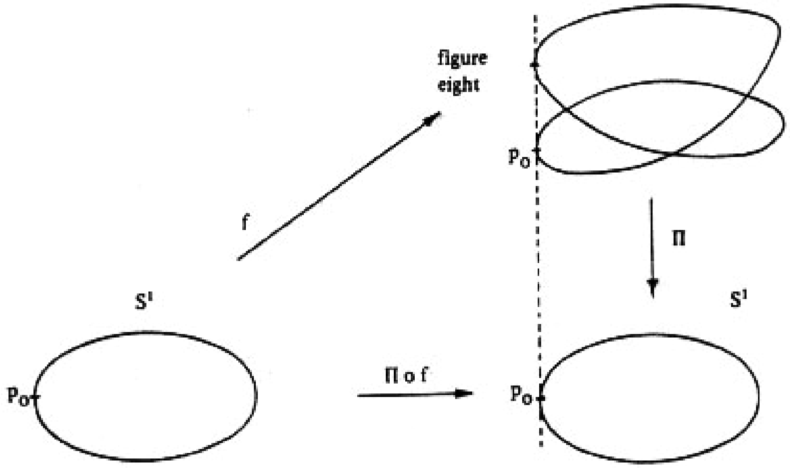


Fig. 1

dimensional choice space. The concept of degree for a map between circles will suffice to prove this special case of the results.

The formal definition of degree of a map between spheres of any dimension is as follows. Let S^n be the nth dimensional sphere in R^{n+1}, and $f:S^n \to S^n$ a continuous map. Let $H_n(S^n)$ denote the nth dimensional singular homology group of S^n with integer coefficients. For further topological definitions, see for instance Spanier (1966). The *degree of the map f*, denoted $\deg(f)$, is defined as the unique integer such that

$$f_n^*(z) = (\deg f)z,$$

where z is a generator of $H_n(S^n)$ and f_n^* is the map induced by the function f at the nth homology level,

$$f_n^*: H_n(S^n) \to H_n(S^n).$$

3. An example

Here we shall study a particular example of the theorem proven in the next section: the case of decisive majority rules when there are three voters and the choice space X is of dimension 2. We shall prove they are necessarily unstable.

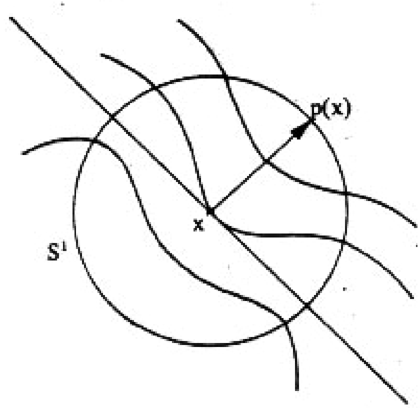


Fig. 2. $p(x)$ is the normal direction of the indifference surface of the preference p at x.

Given a choice x in X, say, the center of the circle in fig. 2, each preference p on X, by definition, determines a unique vector z in S^1, given by the intersection of the normal $p(x)$ to the indifference surface of the preference at x, with the circle S^1.

When the space of preferences P is given the topology of C^1 uniform convergence of vector fields on X, this correspondence $p \to z$ determines a continuous map Γ from the space of preferences to the circle, $\Gamma: P \to S^1$.

Reciprocally, for each z in S^1, we can define continuously a preference p in P such that z is the intersection with S^1 of the vector $p(x)$, the normal to its indifference surface at the choice x. For instance, one can choose the linear preference with all its indifference surfaces normal to $p(x)$. This determines in turn a map $\lambda: S^1 \to P$, which is also continuous. The composition of these two maps $\Gamma \circ \lambda$, is a continuous map from S^1 to S^1, actually the identity map on S^1.

It is easy to check that the properties of Pareto and decisive majorities of the social aggregation rule $\phi: P^3 \to P$ (if it exists) will be inherited by the map ψ defined by the following diagram:

$$\begin{array}{ccc} P^3 & \xrightarrow{\phi} & P \\ {\scriptstyle \lambda^3}\uparrow & & \downarrow{\scriptstyle \Gamma} \\ (S^1)^3 & \xrightarrow[\psi]{} & S^1 \end{array}$$

i.e. $\psi(p_1, p_2, p_3) = \Gamma \circ \phi(\lambda(p_1), \lambda(p_2), \lambda(p_3))$, for all $(p_1, p_2, p_3) \in (S^1)^3$.

Therefore we shall now restrict ourselves to the study of such a map ψ that satisfies continuity, Pareto and decisive majority conditions. We shall prove that such a map cannot exist. This in turn will imply that the corresponding social aggregation rule $\phi: P^3 \to P$ that would induce the map ψ cannot exist either, i.e., ϕ cannot be simultaneously continuous and a decisive majority rule. Therefore any decisive majority rule will be necessarily structurally unstable.

We now discuss the Pareto property of the aggregation rule. Because we are concerned here with the case when there are two homogeneous groups of voters, i.e., when the three voters have among them only two preferences, we shall consider the case of two preferences.

Given two preferences p_1, p_2 and a choice x in X, the Pareto property implies that the social aggregation rule ϕ must have at the choice x an indifference surface whose normal $p(x)$ is contained in the cone determined by the shaded area in fig. 3. This is because the Pareto condition implies that any utility function u representing p must increase in the directions that both u_1 and u_2 increase, where u_1 and u_2 are any utility representations of p_1 and p_2, respectively. Therefore the vector $p(x)$ must be a convex combination of $p_1(x)$ and $p_2(x)$ and this determines the set of directions in the shaded cone. In particular, when $p_1(x) = p_2(x)$, $p(x)$ must be equal to $p_1(x)$.

We now examine geometrically the property of decisive majorities. We assume that there are two homogeneous groups, e.g. that voters 1 and 2 have the same preference denoted p_1, and voter 3 has a different preference, denoted p_2. Assume that p_1 is fixed, and that p_2 is allowed to vary over S^1. The outcome of the social rule will then determine a map from S^1 to S^1, given by the normal at x of the social preference as p_2 varies over S^1, i.e., $\psi(p_1, p_1, \cdot): S^1 \to S^1$. The decisive majority condition implies that when p_2 is the antipodal of p_1, the outcome $\psi(p_1, p_1, p_2)$ must be equal to p_1.

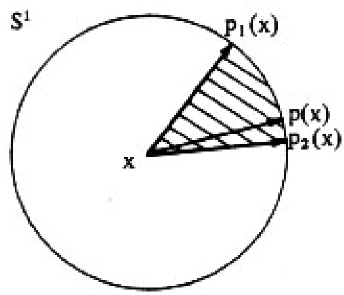


Fig. 3. A geometrical representation of the Pareto property for 2 homogeneous groups of 3 voters, on a 2-dimensional choice space.

We shall now show that the Pareto and the decisive majority conditions considered together imply that the image of S^1 under $\psi(p_1, p_1, \cdot)$ is contained in a strict subset of S^1. This is because by the decisive majority condition, when $\bar{p}_1$ is the antipodal of p_1, it follows that $\psi(p_1, p_1, \bar{p}_1) = p_1$; if p is contained instead in the complement of the set $\{\bar{p}_1\}$, i.e., $p \in S^1 - \{\bar{p}_1\}$, then by the Pareto property, $\psi(p_1, p_1, p)$ must be also in $S^1 - \{\bar{p}_1\}$. Since ψ is continuous, the image of $\psi(p_1, p_1, \cdot)$ must therefore be a strict subset of S^1, that doesn't contain $\bar{p}_1$, the antipodal of p_1.

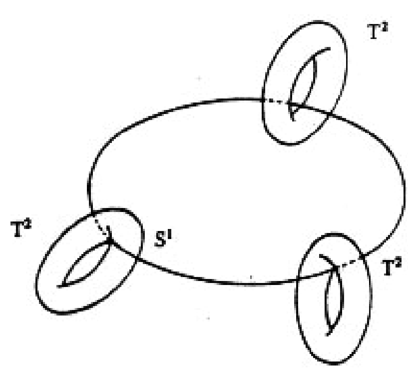


Fig. 4. The figure represents the space of profiles of 3 voters at a fixed choice x. T^2 are 2-dimensional tori $T^2 = S^1 \times S^1$; T^3 represents the 3-dimensional torus $S^1 \times S^1 \times S^1$. Given a fixed choice x in X, a profile of any triple of preferences can be represented as a point in T^3.

We shall now study the implication of the above in terms of the degree of the map ψ on certain submanifolds of the choice space. Let $(S^1)^3$ be denoted T^3, the 3-dimensional torus. If $(S^1)^2$ is the 2-dimensional torus T^2 represented in fig. 4, then T^3 can be represented by the product of T^2 with S^1. As the 3-dimensional manifold T^3 cannot be represented in R^3, fig. 4 is only intended as a suggestive picture of it.

Let D be the diagonal of T^3, $D=\{(p_1,p_2,p_3)\in T^3$ such that $p_1=p_2=p_3\}$.

Consider the diagonal map d,

$$S^1 \xrightarrow{d} T^3,$$

defined by $d(p)=(p,p,p)$; d maps S^1 onto D. $\psi\circ d$ is therefore a map from S^1 to S^1, and we can study its degree. By the Pareto property, $\psi\circ d$ will have degree 1, since Pareto implies that $\psi(p,p,p)=p$ for all p in S^1, so that ψ wraps S^1 around S^1 exactly once. Let p_0 be a fixed point in S^1. We define I_1, an inclusion map of S^1 into T^3, by $I_1(p)=(p,p_0,p_0)\in T^3$. Similarly define I_2 and I_3 by $I_2(p)=(p_0,p,p_0)$ and $I_3(p)=(p_0,p_0,p)$, respectively. The composition $\psi\circ I_i$ maps S^1 into S^1 as indicated in the diagram

$$\begin{array}{ccc} T^3=S^1\times S^1\times S^1 & \xrightarrow{\psi} & S^1 \\ \uparrow I_i & \nearrow \psi\circ I_i & \\ S^1 & & \end{array}$$

Therefore we can study the degree of $\psi\circ I_i$ for $i=1,2,3$. Now, since ψ satisfies the Pareto and decisive majority conditions, as shown above, $\psi\circ I_i(S^1)$ is a strict subset of S^1. Therefore the degree of $\psi\circ I_i$ is zero for all i since the image of $\psi\circ I_i$ does not cover S^1. Define now the map $I:S^1\to T^3$ by

$$I(p)=I_1(3p) \quad \text{iff} \quad p\in[0,2\pi/3],$$

$$I(p)=I_2(3p) \quad \text{iff} \quad p\in[2\pi/3,4\pi/3],$$

$$I(p)=I_3(3p) \quad \text{iff} \quad p\in[4\pi/3,2\pi].$$

I is a continuous map from S^1 to T^3. It maps S^1 onto the three circles which are crossed in the torus T^3 of fig. 5, or, equivalently, into the double figure eight on the right, with the points p_0 identified. Let $\Pi_i:T^3\to S^1$, be the projection of T^3 onto its ith coordinate. Then the composition of I with each Π_i $(i=1,2,3)$ has degree 1, since it wraps around S^1 exactly once. Similarly, the diagonal map d composed with each Π_i has degree 1.

We now use the following:

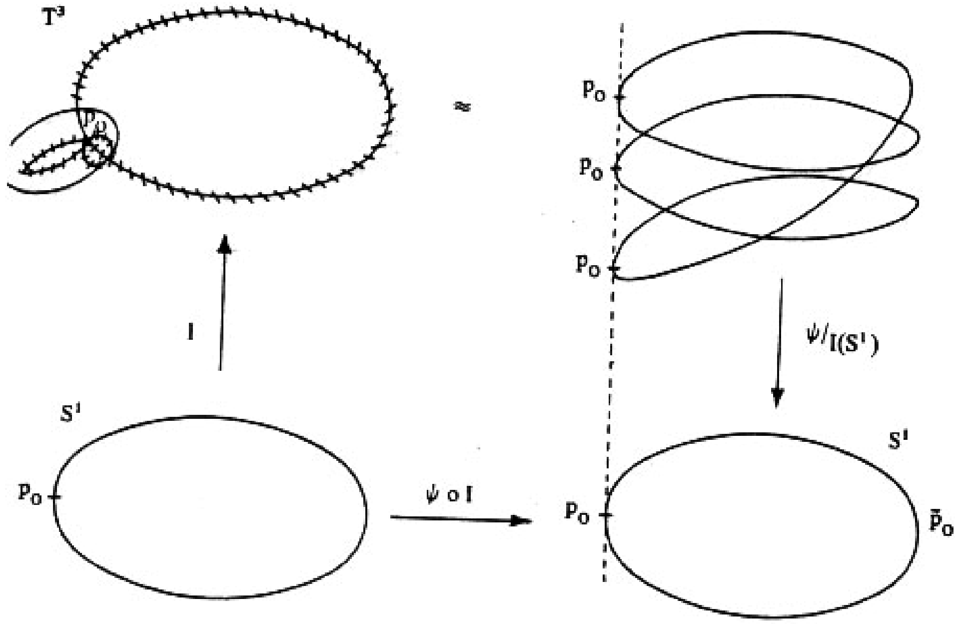


Fig. 5

Lemma 1. Let f and g be two continuous maps from S^1 to T^3. Then f is a continuous deformation of g if and only if the composition map $\Pi_i \circ f$ has the same degree as the composition map $\Pi_i \circ g$ for $i=1,2,3$, where $\Pi_i : T^3 \to S^1$ is the projection map that assigns to a point in T^3 its ith *coordinate.*

A more general version of this lemma will be proven in Lemma 2 of the next section. From Lemma 1 and the fact that for all $i=1,2,3$, $\Pi_i \circ d$ and $\Pi_i \circ I$ have the same degreeon S^1, it follows that the maps d and I are continuous deformations of each other.

Now, the first homotopy group of T^3, $\Pi_1(T^3, p_0)$ can be intuitively described as a group of classes of maps from S^1 to T^3, each class consisting of continuous maps which are all continuous deformations of each other. Therefore, from Lemma 1, d and I are in the smae class within $\Pi_1(T^3, p_0)$; this is indicated by

$$[d]=[I].$$

It follows therefore that the homotopy class of the composition map $\psi \circ d$ is equal to the homotopy class of the composition map $\psi \circ I$ [see e.g. Spanier (1966)], i.e.,

$$[\psi \circ d]=[\psi \circ I].$$

Since both $\psi \circ d$ and $\psi \circ I$ map S^1 into S^1 it follows by the definition of degree that

$$\deg(\psi \circ d) = \deg(\psi \circ I).$$

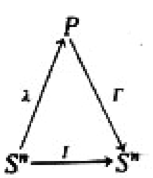
However, as we saw above, the Pareto and the decisive majority conditions together imply that $\deg(\psi \circ I)$ is zero since the image of the map ψ restricted to the set $I(S^1)$ (i.e., restricted to the union of the three circles of the top of fig. 5) does not cover S^1, because it never assumes the value $\bar{p}_0$ antipodal to p_0. Therefore we have obtained a contradiction because as seen above $\deg(\psi \circ d) = 1$. Pareto and decision majority conditions cannot simultaneously hold when the aggregation rule ϕ, and therefore the map ψ, are continuous. Therefore any decisive majority with three voters, in a 2-dimensional choice space is necessarily structurally unstable. This completes our example.

4. Instability of decisive majority rules: A topological degree theorem

We now prove the main result on instability of decisive majority rules for any finite number of voters, and any dimension $n+1 \geqq 2$ of the choice space X. The proof given in this section uses definitions and results of algebraic topology; for further reference, see for instance Spanier (1966).

Theorem. *Any decisive majority rule $\phi: P^k \to P$ is unstable.*

Proof. Let x be a choice in X, and let S^n be the unit sphere in R^{n+1}. As seen in section 3, given a chart for X at x, each preference p in P determines uniquely a point z in S^n, the intersection of the unit vector normal to the indifference surface of p at x, $p(x)$, with S^n. This determines a map Γ from the space of preferences P into S^n, which is continuous by the choice of topology in P. The map Γ can be chosen to be continuous and *onto* S^n. We can also define a continuous map $\lambda: S^n \to P$, by giving for each z in S^n a linear preference on X, with normal vector $p(x)$, such that $z = p(x) \cap S^n$. Therefore we have



(1)

The map defined by making the diagram (1) commutative, i.e., $\Gamma \circ \lambda$, is the identity on S^n.

Assume that a map $\phi = P^k \to P$ exists, satisfying the Pareto and decisive majority conditions. Then we can define a map ψ by the diagram

$$\begin{array}{ccc} P^k & \xrightarrow{\phi} & P \\ \lambda^k \uparrow & & \downarrow \Gamma \\ (S^n)^k & \xrightarrow{\psi} & S^n \end{array} \tag{2}$$

i.e., $\psi(z_1,\ldots,z_k) = \Gamma(\phi(\lambda(z_1),\ldots,\lambda(z_k)))$ for all $(z_1,\ldots,z_k)$ in $(S^n)^k$. ψ is continuous since it is the composition of continuous maps. One can check that ψ satisfies also the Pareto and decisive majority conditions on its domain.

It is useful to visualize the conditions in geometrical terms. The Pareto condition implies geometrically that the outcome vector is contained in a cone generated by the individual normal vectors at choice x. In the particular example illustrated in fig. 6 there is a 3-dimensional choice space X and three voters. Fig. 7 is the case of two voters and a 3-dimensional choice space.

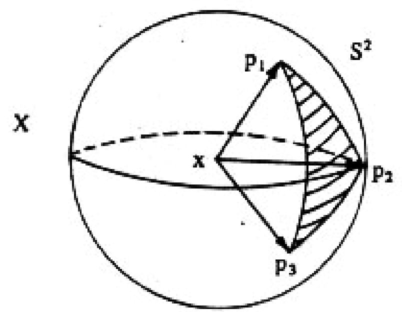


Fig. 6. If the social rule is Pareto, the outcome corresponding to p_1, p_2, p_3 must be contained in the shaded area of S^2.

Because of the previous construction, we can reduce the problem of non-existence of the social aggregation rule ϕ to one of non-existence of the map ψ on the product of spheres, satisfying the Pareto and decisive majority conditions. We shall study this next.

In the following we examine the topological degree of the map ψ when restricted to the diagonal D of the space $(S^n)^k$,

$$D = \{(p_1,\ldots,p_k): p_i \in S^n, p_i = p_j, \forall\, i,j\}.$$

First note that the Pareto condition on ψ implies that when all vectors in S^n are the same, say all equal to p, then $\psi(p,\ldots,p) = p$.

Let $d: S^n \to (S^n)^k$ be defined by $d(p) = (p,\ldots,p)$.

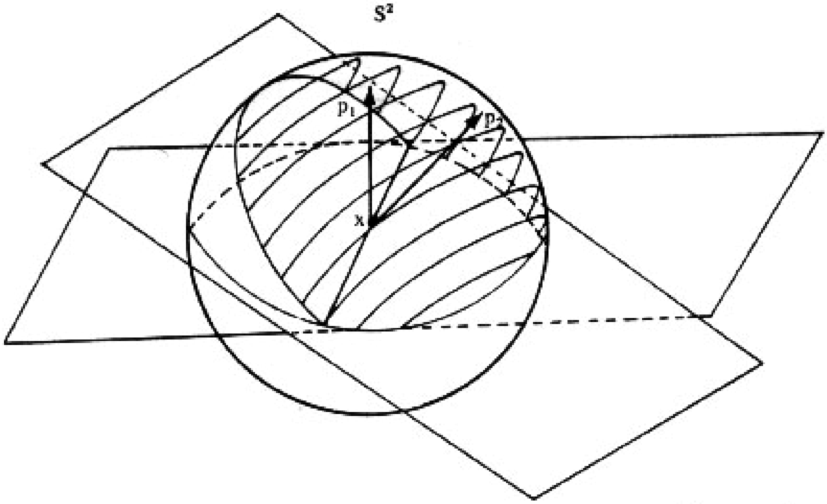


Fig. 7. The Pareto condition with 2 voters in a 3-dimensional choice space. The voters' normals to the indifference subspaces are p_1 and p_2; the Pareto condition requires that the outcome has a positive inner product with all vectors within the shaded area.

It follows that the Pareto condition implies $\deg(\psi \circ d) = 1$.

Let $\pi_n(S^n)$ be the nth homotopy group of S^n.

Define the jth inclusion map $I_j : S^n \to S^n \times \ldots \times S^n$ by

$$I_j(p) = (p_0, \ldots, p_0, \overbrace{p}^{j\text{th place}}, p_0, \ldots, p_0).$$

Let $\Pi_j : (S^n)^k \to S^n$ be the projection map onto the jth coordinate, i.e., $\Pi_j(p_1, \ldots, p_k) = p_j$. Then the composition $\Pi_j \;\; I_j$ is the identity map of S^n.

We shall now use the following lemma:

Lemma 2. Let $\alpha, \beta \in \Pi_n((S^n)^k)$. *Then* $\alpha = \beta$ *if and only if* $\Pi_j^* \alpha = \Pi_j^* \beta$ *for* $j = 1, \ldots, k$, *where*

$$\Pi_j^* : \Pi_n((S^n)^k) \to \Pi_n(S^n)$$

is the homomorphism induced by Π_j *at the homotopy level.*

Proof. The group $\Pi_n((S^n)^k)$ is isomorphic to $\oplus_1^k \Pi_n(S^n)$; see Spanier (1966, p. 418, B4). The projections Π_j induce projections Π_j^* at the group level, i.e., $\Pi_j^* \circ I_j^* = id^*$, where I_j^* is the map induced at the homotopy level by the inclusion map I_j. Therefore, since an element in the group $\oplus_1^k \Pi_n(S^n)$ is identified by its projections, the result follows.

From Lemma 2 it follows that the homotopy class of the diagonal map $d: S^n \to S^n \times \ldots \times S^n$ is the 'sum' of the homotopy class of the I_j maps, namely,

$$[d] = [I_1] + \ldots + [I_k]. \tag{3}$$

where [ ] denotes homotopy class in $\Pi_n((S^n)^k)$. It follows from (3) that

$$[\psi \circ d] = [\psi \circ I], \tag{4}$$

where I is a function $I: S^n \to (S^n)^k$ such that $[I]$ is the element of $\Pi_n((S^n)^k)$ that represents $\sum_j [I_j]$. Therefore, by (3) and (4),

$$\deg(\psi \circ d) = \deg(\psi \circ I), \tag{5}$$

since the degree of a map from S^n to S^n is a homotopy invariant, and yields an isomorphism $\Pi_n(S^n) \approx Z$, where Z is the additive group of integers.

We now study the degree of $\psi \circ I$ when ψ satisfies the Pareto and decisive majority conditions. As seen in the example of section 3, these two conditions when taken together, imply that the image of ψ restricted to $I_j(S^n)$ will never assume the value $\bar{p}_0$ antipodal to p_0, for all j. This is proven as follows. On the set $I_j(S^n)$, all voters except for the jth have the same preference at the choice x, namely p_0, while the jth preference may vary over its whole domain, thus describing in terms of its preferred direction at x, a sphere S^n. By the decisive majorities condition, when the jth voter's preferred direction is the antipodal of p_0, $\bar{p}_0$, the outcome

$$\psi(p_0, \ldots, \overbrace{\bar{p}_0}^{j\text{th place}}, p_0, \ldots, p_0)$$

must be contained in the set $S^n - \{\bar{p}_0\}$. Moreover, when the preferred direction of the jth voter is in $S^n - \{\bar{p}_0\}$, the Pareto condition implies that the outcome must also be contained in $S^n - \{\bar{p}_0\}$. Therefore the image of ψ restricted on $I_j(S^n)$ is a strict subset of S^n which does not contain $\bar{p}_0$. Since I can be chosen so that

$$I(S^n) \subset \bigcup_j I_j(S^n), \quad \text{this implies} \quad \deg(\psi \circ I) = 0. \tag{6}$$

From (5) and (6), and the fact that the Pareto condition implies instead $\deg(\psi \circ d) = 1$, we obtain a contradiction. A continuous map $\psi: S^n \times \ldots \times S^n \to S^n$ satisfying both Pareto and decisive majority conditions cannot exist. Therefore any decisive majority rule ϕ must necessarily be structurally unstable. This completes the proof of the theorem.

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