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EXISTENCE OF OPTIMAL SAVINGS POLICIES WITH IMPERFECT INFORMATION AND NON-CONVEXITIES*

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This paper studies the existence of optimal savings paths in models with imperfect information. The returns on savings or on investment are uncertain and the models may have infinite horizons. The results yield existence when both utilities and the distributions of the random rates of return may vary through time. The cases studied here also include nontrivial non-convexities on both the utility functions (which are not necessarily bounded) and the feasible sets of consumption paths which may appear because of incomplete information, externalities or increasing returns. Existence is also established when there is learning about the value that determines the distribution of returns, updated according to previous realizations. The techniques use non-linear functional analysis. The results are obtained by proving compactness/continuity theorems of certain non-linear operators on Hilbert spaces and other function spaces, and are based on previous results in Chichilnisky (1977, 1981).

1. Introduction

Individual optimal behavior under uncertainty has been extensively studied in the literature. However, a problem that remains to be examined is the existence of optimal savings paths in models with imperfect information on the uncertain returns on savings or investment, and with an infinite time horizon. Even in the special cases where uncertainty is given by a random rate of return with a known distribution, identically distributed through time, existence of paths that maximize the objective function is not easy to obtain. In Levhari–Srinivasan (1969), for example, the objective function the individual maximizes is

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c(t))\right], \tag{1}$$

where E is the expected value operator, the expectation being over the joint distribution of the random variables $c(t)$ denoting consumption at time t, and $\beta \in (0,1)$ is a discount factor. In this model the distribution of $c(t)$ is

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known and is constant through time. This distribution is derived from that of the random rate of returns θ through the stochastic constraints

$$\omega(t+1)=(\omega(t)-c(t))\theta, \qquad \theta,\ \omega_0 \text{ given}, \tag{2}$$

where ω_t denotes wealth at time t. A first problem in proving existence of a solution is that the expression (1) is not necessarily well defined. The sum may not be convergent, and even if it is convergent, the $\max E[\sum_{t=0}^{\infty} \beta^t u(c(t))]$ may fail to exist.[1] These types of problems become more complex in models where there is, in addition, imperfect information about the distribution of the rate of returns of savings. In these cases, the random variable $\theta(t)$ has an unknown distribution, for instance, a distribution depending on an unknown parameter $\bar{\theta}$, and the distributions of the random variables $\theta(t)$ and $c(t)$ typically change through time. For example, one can assume that the successive observations of $\theta(t)$ are independent drawings, and that at each realization of the parameter $\theta(t)$ the individual prior is transformed into a posterior, through a learning process [or a Bayesian rule as in Scarf (1959)]. One case is that of increasing information, i.e., where the information about the value of the unknown parameter $\bar{\theta}$ converges to certainty. In other cases, the underlying parameter is not constant, but itself changing through time in accordance with some stochastic process.[2]

In this note we generalize and extend existing results on optimal savings under uncertainty to the more general models just discussed. We build up a technique that allows us to prove existence of optimal savings paths in models where the distribution of the random variables changes through time, in a fashion that may depend on the previous realizations of these random variables (e.g. when there is learning) and under quite general specifications of utilities and technologies. The plan of this paper is to build from special cases to more general ones — indeed, the general results were motivated by special ones.

The models we consider here in the more general cases, and the results obtained, generalize in particular those of Levhari–Srinivasan (1969) and Mirlees (1965). They include cases with imperfect information on returns and time dependent utilities. In addition, existence is also proven with non-trivial non-convexities on both utilities and feasible sets of consumption paths, and when utilities are not necessarily bounded. Such non-convexities appear typically in these types of models associated with incomplete information or increasing returns.

[1]Given the fact that (1) may not be well defined, Levhari and Srinivasan use an overtaking criterion instead of the one formulated initially at least in some parts of their paper. In our case we study conditions under which (1) is well defined, and prove existence of $\max E[\sum \beta^t u(c(t))]$ as a special case of the results.

[2]For instance, a random walk as discussed in stock market exchange models.

In section 2, we consider cases where both the utilities and the distributions of the random rates of return may vary through time. We give conditions for continuity and boundedness of the relevant operators in Lemmas 1 and 2.

In section 3, more complex cases of imperfect information are considered: in Theorem 3 the distribution function of the random returns $\theta(t)$ is assumed to be time-dependent and uncertain; in Theorem 4 there is also learning about the value of a parameter, denoted $\bar{\theta}(t)$, that determines the distribution of $\theta(t)$. At each time period the distribution of the random variable is updated according to the previous realizations of that variable, i.e.,

$$\bar{\theta}(t+1)=L(\bar{\theta}(t),\theta(t),t),$$

where L is a learning function. We also consider cases where the learning about the distribution of $\theta(t)$ is allowed to accumulate according to all previous realizations of the variable, i.e.,

$$\bar{\theta}(t+1)=L(\bar{\theta}(t),\theta(1),\ldots,\theta(t),t).$$

Further cases where information converges to certainty are considered, i.e., when $\bar{\theta}(t)\to\bar{\theta}$ as $t\to\infty$, where $\bar{\theta}(t)$ is the value of the parameter determining the distribution of $\theta(t)$. In section 3, the models allow for production, i.e., the constraint (2) is replaced by the more general constraint

$$\omega(t+1)=f(\theta(t),\omega(t)-c(t),t), \tag{3}$$

where f is a production function.[3] Theorems 3 and 4 in section 3 prove the existence of optimal savings paths under imperfect information about $\theta(t)$ and learning, as discussed above, including cases where there are non-trivial non-convexities both on the utilities and on the technology (the function f), and utilities are not necessarily bounded.

The proofs use techniques of non-linear functional analysis in certain Hilbert spaces. Related methods were used in Chichilnisky (1977, 1981) and Chichilnisky–Kalman (1980) for the study of efficiency and optimality in growth models with an infinite horizon. However, in this case extension of the results of those papers is needed: for instance, the complexities that appear due to the underlying uncertainty and/or imperfect information considered here require us to work on spaces of infinite sequences of function spaces. The more technical aspects of the results obtained here are contained in the appendix.

[3] (3) contains (2) as a special case.

2. A special case of the model and some results

Here the individual policy is to maximize a discounted sum of time-dependent utilities

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c(t), t)\right], \qquad \beta \in (0, 1), \tag{4}$$

where E is the expected value operator, subject to the stochastic constraints

$$\omega(t+1) = (\omega(t) - c(t)) \cdot \theta(t), \qquad \theta(0) \text{ given}, \tag{5}$$

where $\theta(t) \in [0, \infty)$ is a random variable with a time-dependent density given by $F(t)$, representing returns on savings. The non-negativity constraints,

$$\omega(t) \geqq c(t) \geqq 0, \qquad \text{for all } t, \tag{6}$$

are also required.

We assume $F(t)$ is a continuous bounded function for each t, and that $u(0, t) = 0$ for all t. The following results generalize the results of Levhari–Srinivasan (1969) with respect to the boundedness of the expectation operator (4), also extending them to more general models where utilities and distributions of returns on savings are not necessarily stationary through time.

We first need some definitions and assumptions. Let $u(x, t)$ represent instantaneous utility at time t, $u: R^+ \times R^+ \to R^+$, and let $\theta(t)$ be independently distributed throug time. Assume that $u(x, t)$ satisfies Caratheodory conditions[4] and that, for all t,

$$0 < \beta E(\theta(t), t) < 1 - \varepsilon, \qquad \varepsilon \in (0, 1), \tag{7}$$

where $E(\theta(t), t)$ is the expected value of $\theta(t)$ with respect to its density $F(t)$; or, in the special case that $F(t) \equiv F_0$, i.e., when the distribution of θ is stationary,

$$0 < \beta E\theta < 1. \tag{8}$$

(This latter condition is also assumed by Levhari–Srinivasan.)

Let H_β be the Hilbert space of paths defined as the space of all sequences $\{e_t\}_{t=0,1,\dots}$, with

$$\sum_{t=0}^{\infty} \beta^t |e_t|^2 < \infty. \tag{9}$$

[4] $u: R^+ \times R^+ \to R^-$ is said to satisfy *Caratheodory conditions* if $u(x, t)$ is continuous with respect to x for almost all t in R^+, and measurable with respect to t for all values of x in R^-.

While we assume that consumption is one-dimensional for simplicity of the exposition, all the results given here are valid for $u: R^{n+} \times R^+ \to R^+$, an n-dimensional consumption vector.

The square root of (9) defines the $\|\cdot\|_\beta$ norm of $\{e_t\}$, denoted $\|\{e_t\}\|_\beta$. H_β is the Hilbert space l_2 with a bounded measure on $[0,\infty)$ given by the density function[5] $\beta(t)=\beta^t$, $t\in[0,\infty)$. For any initial wealth ω_0, a consumption path $\{c(t)\}$, $t=0,1,\dots$, is called *feasible* if there exists a sequence $\{\omega(t)\}$, $t=0,1,\dots$, $\omega(0)=\omega_0$, such that $\{c(t)\}$ and $\{\omega(t)\}$ satisfy (5) and (6) above. Similarly, one defines feasible wealth paths $\{\omega(t)\}$. The following lemma is used in the proof of Lemma 2 and in the existence of the next section:

Lemma 1. For any initial wealth ω_0 and feasible consumption path $c(t)$ under conditions (7) or (8), the set of all sequences of expected values of consumption $\{E(c(t),t)\}$, is contained in H_β. Furthermore, the operator $Y:H_\beta\to R^+$, defined by

$$Y(E(c(t),t))=\sum_{t=0}^{\infty}\beta^t u(E(c(t),t)),$$

is $\|\cdot\|_\beta$ continuous as a function of the variable $\{E(c(t),t)\}$ if and only if the utility u satisfies the following condition:

(C.1) $$|u(x,t)|\leqq a(t)+b|x|^2,$$

where b is a positive constant, $a(t)\geqq 0$, and $\sum_{t=0}^{\infty}\beta^t a(t)<\infty$.[6] Hence, in particular, when (C.1) is satisfied, the sum $\sum_{t=0}^{\infty}\beta^t u(E(c(t),t))$ is bounded.

Proof. Let $\overline{\omega(t)}=\omega_0\prod_{s=0}^{t-1}\theta(s)$.

Any feasible $\{\omega(t)\}$ satisfies $\omega(t)\leqq\overline{\omega(t)}$ for all t. Note that, since $\omega(t)\geqq c(t)\geqq 0$, by assumption (6),

$$\begin{aligned} E(c(t),t)&\leqq E(\omega(t),t)\\ &\leqq E\omega_0\prod_{s=0}^{t-1}(\theta(s),s)\\ &=\omega_0\prod_{s=0}^{t-1}\int\theta(s)\,\mathrm{d}F(s)\\ &\leqq\frac{\omega_0}{\beta}(1-\varepsilon)^t, \end{aligned}\tag{10}$$

[5]A density function is a positive real valued function on the set of integers $[0,\infty]$ which is used as a measure to define the summability, or integrability of real valued functions on $[0,\infty]$ in l_2, as in (9) above.

[6]Note that condition (C.1) does not imply the existence of a function $U(x)$ that provides a uniform bound (for all t) to the $u(x,t)$'s, since the functions $a(t)$ may increase without bound in the space H_β as t goes to infinity.

since the $\theta(t)$'s are independently distributed through time and in view of (7) above. In particular, if $\theta(t)$ is identically distributed through time, then $0<\beta E\theta<1$ suffices to show that

$$\sum_{t=0}^{\infty} \beta^t E(c(t),t)<\infty,$$

and also

$$\sum_{t=0}^{\infty} \beta^t [E(c(t),t)]^2<\infty.$$

By (10), the sequence $\{E(c(t),t)\}_{t=0,1,\ldots}$ is in H_β, for all feasible paths $\{c(t)\}$, and for any initial ω_0. We now study the continuity of Y.

Denote $\{E(c(t),t)\}$ by $\{e_t\}$ or e, and let $d=\{d(t)\}=\{\beta^t e(t)\}$. Then the map

$$Y: H_\beta \to R^+,$$

given by

$$e \xrightarrow{Y} Y(E(c(t),t))=\sum_{t=0}^{\infty} \beta^t u(E(c(t),t)),$$

is $\|\cdot\|_\beta$ continuous if and only if the map Z, given by

$$d \xrightarrow{Z} \{\beta^t u(\beta^{-t} d(t))\},$$

is continuous from $l_2[0,\infty)$ to $l_1[0,\infty)$,[7] since

$$\{y(t)\} \xrightarrow{\|\cdot\|_\beta} y^0(t),$$

if and only if $\{\beta^t y(t)\}\to\{\beta^t y^0(t)\}$ in l_2. By Krasnosel'skii (1964, theorems 2.1, 2.3, pp. 22–28, and remarks, p. 28), a necessary and sufficient condition for Z to be continuous from l_2 to l_1 is that

$$|\beta^t u(\beta^{-t} d(t))| \leqq g(t)+\alpha|d|^2,$$

where $g(t)\in l_1$, $g(t)\geqq 0$ for all t, and α is a positive constant. This is equivalent to

$$|u(e(t))| \leqq a(t)+b|e|^2,$$

[7] $l_1[0,\infty)$ is the Banach space of all sequences $\{a(t)\}$ with $\sum_{t\geq 0}|a(t)|<\infty$, and $l_2[0,\infty)$ is the Hilbert space of all sequences $\{a(t)\}$ with $\sum_{t\geq 0}|a(t)|^2<\infty$.

where $a(t) \geq 0$ for every t, $\sum_{t=0}^{\infty} \beta^t a(t) < \infty$ and $b \geq 0$. This completes the proof.

The next application of Lemma 1 generalizes the results of Levhari–Srinivasan with respect to the boundedness of the expectation operator in (4); here the utilities are not necessarily bounded, and both the utilities and the distributions of the random variables may change through time.

Lemma 2. If u is strictly concave, then under condition (C.1) *of Lemma 1,*

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c(t),t)\right]$$

is always bounded, for any feasible consumption path $\{c(t)\}$ with initial condition ω_0 in the above model, where utilities and the distribution of the random rate of return may vary through time. Furthermore,

$$\sup_{\{c(t)\}} E\left[\sum_{t=0}^{\infty} \beta^t u(c(t),t)\right]$$

is also bounded.

Proof. This follows from an application of Lemma 1. Since u is strictly concave, by Jensen's inequality and the results of Lemma 1,

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c(t),t)\right] \leq \sum_{t=0}^{\infty} \beta^t u(E(c(t),t)),$$

which is bounded.[8] From condition (6) and Lemma 1, all feasible sequences $\{E(c(t),t)\}$ are uniformly bounded above. Lemma 2 of Chichilnisky (1977) implies that these feasible sequences form a pre-compact set in H_β. The continuity of Y, proven in Lemma 1, completes the proof.

3. The model and existence of solutions

We now turn to the more general model and the existence results. The individual's policy is to maximize

$$W(c(t)) = E\left[\sum_{t=0}^{\infty} \beta^t u(c(t),t)\right], \qquad \beta \in (0,1), \tag{11}$$

[8]Note that even if all $u(c(t),t)$ were uniformly bounded Lemma 2 is not valid when the u's are not concave, since Jensen's inequality does not apply. Theorem 3, below, extends Lemma 2 to cases where u is not concave.

subject to the stochastic constraints[9]

$$\omega(t+1)=f(\theta(t),\omega(t)-c(t),t), \tag{12}$$

where $\theta(t)$ is a random variable with a distribution identified by a parameter $\bar{\theta}(t)$, f is a continuous function representing the technology and the non-negative constraints,

$$\omega(t)\geqq c(t)\geqq 0, \qquad \omega_0 \text{ given}, \tag{13}$$

are satisfied.

In general, $\theta(t)$ is a random variable whose distribution changes through time. A special case is where there is learning about the value of the parameter $\bar{\theta}(t)$ which determines the distribution of $\theta(t)$. This can be formalized by the constraint

$$\bar{\theta}(t+1)=L(\bar{\theta}(t),\theta(t),t), \tag{14}$$

i.e., $\bar{\theta}(t+1)$ depends on $\bar{\theta}(t)$ and the realized value of the parameter at time t, $\theta(t)$. In addition, if for all t, $\theta(t)$ is assumed to have an unknown but fixed distribution identified by a parameter $\bar{\theta}$, information is increasing, by definition, when

$$\bar{\theta}(t)\rightarrow\bar{\theta} \quad \text{as} \quad t\rightarrow\infty, \tag{15}$$

In these latter cases, W of (11) becomes a conditional expectation operator,

$$E\left[\sum_{t=0}^{\infty} u(c(t),t)\,\middle|\,\bar{\theta}_0\right], \tag{16}$$

where $\bar{\theta}_0$ is the prior value of the parameter at time 0, and where the expectation is taken with respect to the joint distribution of the random variables $c(t)$, these distributions being induced by those of the random variables $\theta(t)$ through (12). The distributions of the $\theta(t)$'s satisfy the updating eq. (14).

We first prove existence of an optimal consumption policy in cases where the distribution of the returns on savings and utilities are not stationary (i.e., they are time-dependent), but there is no learning. In these cases, the distributions of the random variables at time t are assumed to be independent of previous realizations of the variables. These results are contained in Theorem 3. Theorem 4 proves existence when the changing distributions are due to change in information, for instance, through a learning process such as that formalized by constraints (14) and (15).

[9]A special case of (12) is $\omega(t+1)=\theta(t)(\omega(t)-c(t))$.

We first need some more definitions. Let $g:R^+\to R$ be a measurable function. Then the $\|\cdot\|_\beta^1$ norm is defined by

$$\|g\|_\beta^1=\int_0^\infty e^{-\beta x}|g(x)|\,dx.$$

The space of all such functions g, for which the norm $\|\cdot\|_\beta^1$ is bounded, is denoted H_β^1. It is a Banach space isomorphic to L_1, with the finite measure induced by the density function $e^{-\beta x}$ on $[0,\infty)$.

Let Γ denote the space of all *sequences of functions* $\{\phi(t)\}_{t=1,\dots}$, where $\phi(t)\in H_\beta^1$ for all t, and satisfying

$$\|\{\phi(t)\}\|_\beta^\Gamma=\sum_{t\geq 0}\beta^t(\|\phi(t)\|_\beta^1)^2<\infty.$$

$(\Gamma,\|\cdot\|_\beta^\Gamma)$ is a Hilbert space isomorphic to the space of maps from the set of positive integers N to H_β^1 with a finite l_2 norm, with respect to a finite measure on the set of integers given by the density function β^t. Let Ω be the space of all maps from N to H_β^1, or sequences $\{\phi(t)\}_{t=0,1,\dots}$, each map $\phi(t)$ in H_β^1, satisfying

$$\|\phi\|_\infty^\Gamma=\sup_{t\in[0,\infty)}(\|\phi(t)\|_\beta^1)<\infty.$$

Ω is a Banach space of infinite sequences of functions in H_β^1. Note that $\Omega\subset\Gamma$. The norm $\|\cdot\|_\infty^\Gamma$ on Ω is also used in the following as an auxiliary topology in the proof of existence.

Since the distributions of the random variables change through time, the consumption policies (i.e., the functions that determine how much to consume as a function of wealth) are also time-dependent. In Levhari–Srinivasan (1969), for example, by contrast, the consumption policies are independent of time, since all the random variables are identically distributed through time. Denote by ψ the set of all *consumption policies* $\phi=\{\phi(t)\}_{t=0,1,\dots}$ satisfying (12) and (13). We shall assume that, for each t, $\phi(t)$ is a measurable function. Then, by Lemma 1, $\phi(t)\in H_\beta^1$ for all t. We thus can assume that $\psi\subset\Gamma$. Note that the set ψ is closed coordinate-wise, i.e., for all t, if $\phi^\alpha(t)\to\phi^0(t)$, then $\phi^0(t)\in H_\beta^1$ satisfies the conditions (12) and (13) that define the set ψ. By definition of the topology of ψ (as a subset of Γ), this implies in turn that ψ is closed as a subset of Γ, since the coordinate-wise topology on ψ is weaker than the $\|\cdot\|_\beta^\Gamma$ norm; see e.g. Dunford and Schwartz (1958).

We now assume that the utilities $u(c,t)$ are increasing in c and the density functions $F(t)$ satisfy $F(t)\leqq e^{-\beta t}N(t)$, where $N(t)\in R^+$, for each t. Note that the choice of the parameter β is flexible within $(0,1)$. We shall further assume the following asymptotic behavior of the expectations and consumption policies:

(C.2) $$E(u(\phi(\omega,t)))\leq a(t)+b(\|\phi\|_{\beta}^{1})^{2}, \qquad \text{for each } t,$$

where

$$\sum_{t\geq 0}\beta^{t}a(t)<\infty, \qquad a(t)\geqq 0 \quad \text{for all } t, \quad \text{and} \quad b\in R^{+}.$$

We can now prove:

Theorem 3. Let u satisfy the conditions of Lemma 1 in section 2, u not necessarily concave. For $\phi=\{\phi(t)\}_{t=0,1,\ldots}$, *a consumption policy in* ψ, *let*

$$W(\phi)=E\left[\sum_{t\geq 0}\beta^{t}u(\phi(\omega(t),t))\right],$$

where $u(\phi(\omega(t),t))$ *represents* $u(\phi(t)(\omega(t),t))$, *and the expectation is taken with respect to the joint distribution of the random variables* $\omega(t)$, *identified by the sequence of parameters*[10] $\{\bar{\theta}(t)\}_{t=0,1,\ldots}$. *Assume that condition* (C.2) *is satisfied for the expectations and the consumption policies. Then W is a* $\|\cdot\|_{\beta}^{\Gamma}$ *continuous map from* ψ *to* R^{+}, *and there exists a sequence of consumption policies* $\phi^{*}=\{\phi^{*}(t)\}_{t=0,1,\ldots}$ *that maximizes* $W(\phi)$ *subject to the constraints* (12) *and* (13). *No concavity assumptions on the technology f in* (12) *or boundedness of the utility u are required.*[11]

Proof. First note that, since $\phi\in\Gamma$, assumption (C.2) implies that the sum

$$\sum_{t=0}^{\infty}\beta^{t}E(u(\phi(\omega(t),t)))$$

is a well defined positive real valued function for all feasible consumption policies ϕ in ψ. Thus, in particular, the expectation operator commutes with summation, i.e.,

$$W(\phi)=\sum_{t=0}^{\infty}\beta^{t}E(u(\phi(\omega(t),t))).$$

Next, note that the function $V:(\psi\times R^{+})\rightarrow R^{+}$, given by $V(\phi,t)=E(u(\phi(t,t)))$ satisfies the Caratheodory conditions: V is continuous with respect to ϕ, and measurable with respect to t for all values of ϕ, by the choice of

[10] If the information converges to certainty, $\bar{\theta}(t)\rightarrow\bar{\theta}$ as $t\rightarrow\infty$.

[11] Boundedness of the utilities u at each time t is also not required, i.e., for each t, $u(c,t)$ is not necessarily a bounded function of c.

topologies and the properties of u (see Lemma 5 in the appendix). Hence the map $\bar{W}:\psi\to l_1$, defined by

$$\bar{W}(\phi)=\{\beta^t Eu(\phi(t)(\omega(t),t))\}_{t=0,1,\ldots},$$

is continuous from ψ to l_1 if condition (C.2) is satisfied (see Lemma 5). But $\bar{W}:\psi\to l_1$ is continuous if and only if $W:\psi\to R$ is continuous; thus (C.2) implies that W is continuous. Finally, note that by the constraint (13) and the fact that, as noted in Lemma 1, $\omega(t)\leqq\overline{\omega(t)}$, the set of feasible consumption policies ψ is $\|\cdot\|_\infty^\Gamma$ bounded in $\Omega\subset\Gamma$: by Lemma 6 in the appendix, then, the set of feasible consumption policies ψ is actually $\|\cdot\|_\infty^\Gamma$ compact. Hence there is a solution to the maximization problem. This completes the proof.

We now turn to the case where the distribution of the random variables is updated at each time period according to the realizations of the variable up to that period. Accordingly, (14) can be rewritten as

$$\bar{\theta}(t+1)=L(\bar{\theta}(0),\theta(1),\ldots,\theta(t),t). \tag{14'}$$

Hence, an optimal policy in this case is a sequence of consumption policy functions $\{\phi(t)\}_{t=0,1,\ldots}$ where, for each t, the policy $\phi(t)$ depends on the past realizations of the variable, $\{\theta(1),\ldots,\theta(t-1)\}$. These, in turn, through the learning process (14) [or (14')] determine, given $\bar{\theta}(0)$, the value of the parameter $\bar{\theta}(t)$ which identifies the (posterior) distribution with respect to which the expectation $E(u(\phi(\omega(t),t)))$ is computed at time t. Thus, in this case, a consumption policy can be thought of as a map from the space of sequences of realizations of the random variables through time, into a sequence of consumption policy functions of the type of those of Theorem 3. Thus, a *strategy of consumption* is now assumed to be an element of Γ^{H_θ}, i.e., a function $\xi:H_\theta\to\Gamma$ assigning to each sequence $\{\theta\}=\{\theta(1),\ldots,\theta(t),\ldots\}$ a sequence $\{\phi(1),\ldots,\phi(t),\ldots\}$, each $\phi(t)$ satisfying (13), where $\phi(t)$ actually depends only on the subsequence $\{\theta(1),\ldots,\theta(t-1)\}$.

An *optimal strategy of consumption* is a function ξ^* such that, for any sequence $\{\theta\}$, ξ^* maximizes the value of

$$W_1(\xi)=E\left[\sum_{t=0}^{\infty}\beta^t u(\xi(\theta)(\omega(t),t))\,|\,\bar{\theta}_0\right],$$

where $u(\xi(\theta)(\omega(t),t))$ represents the utility of the consumption $\phi(t)(\omega(t),t)$ derived from wealth $\omega(t)$, and the consumption policy ϕ where $\phi=\xi(\theta)$, i.e., $\{\phi(t)\}=\{\xi(\theta)(t)\}$. The expectation operator conditional on $\bar{\theta}_0=\bar{\theta}(0)$ is taken with respect to the joint distribution of the random variables through time as given by the updating time dependent process described above. [It is assumed that the parameters $\bar{\theta}(t)$ provide a sufficient statistic of the distribution of $\theta(t)$.] A *feasible strategy of consumption* is an element of the set ψ^{H_β}, i.e., a strategy of consumption such that, for each realization of the random variables $\{\theta\}$, it gives a feasible consumption policy $\psi(\theta)\in\psi$. Let the space Γ^{H_β} be given the product topology, i.e., the topology of coordinate convergence. We assume the following asymptotic behavior of the expectations, utilities, and learning process:

$$\beta^t Eu(\xi(\theta)(\omega(t),t))|\bar{\theta}_0 \leq a(t)+b(\|\xi(\theta)\|_\beta^\Gamma)^2, \tag{C.3}$$
$$a(t)\geqq 0 \quad \text{for all } t, \quad \sum_{t\geq 0}\beta^t a(t)<\infty, \quad b\in R^1.$$

In view of Theorem 3 we can now prove:

Theorem 4. Assume that, for each feasible strategy of consumption ξ in ϕ, condition (C.3) *on the expectations, the utilities and the learning process of* (14′) *is satisfied at each time t, for each sequence $\{\theta\}$ of realizations of the random variables. Also, assume that the conditions on feasible consumption policies and utilities of Theorem 3 are satisfied. No concavity or boundedness of the utilities u or concavity of the technology f in* (12) *is required. Then the map $W_1:\psi^{H_\beta}\to R^+$, defined by*

$$W_1(\xi)=E\left[\sum_{t=0}^{\infty}\beta^t u(\xi(\theta)(\omega(t),t))\,|\,\bar{\theta}_0\right],$$

is well defined and continuous, and there exists an optimal strategy of consumption, i.e., a $\xi^*\in\psi^{H_\beta}$ which maximizes W_1 subject to constraints (12), (13) *and* (14′).

Proof. In view of the choice of topology for ψ^{H_β}, W_1 is continuous on $\xi\in\psi^{H_\beta}\subset\Gamma^{H_\beta}$ if and only if for each $\{\theta\}\in H_\beta$, the map

$$W_1(\xi(\theta)):\psi\to R^+$$

is continuous, since W_1 will be continuous if and only if it is continuous coordinate-wise [see Dunford–Schwartz (1958)]. But $W_1(\xi(\theta))=W(\phi)$, where W is the map defined in Theorem 3 and $\phi=\xi(\theta)$, i.e. $\phi=\phi(t)=\{\xi(\theta)(t)\}$. Hence (C.3)$\Rightarrow W$ is continuous by Theorem 3. Thus, (C.3)$\Rightarrow W_1$ is continuous coordinate-wise, and thus W_1 is continuous. Next note that the assumptions on the utilities and the consumption policies of Theorem 3 imply that ψ is $\|\cdot\|_\beta^\Gamma$ compact. Therefore, by the assumptions of Theorem 3, ψ^{H_β} is compact coordinate-wise. This implies, by Tychonoff's theorem, that ψ^{H_β} is compact in the product topology, which completes the proof.

Appendix

In this appendix we prove two technical results used in the paper.

Lemma 5. Under the conditions of Theorem 3, if $V:(\psi \times R)\to R^{+}$ *is given by* $V(\phi,t)=Eu(\phi(\omega),t)$, $t\geqq 0$, $\phi=\{\phi(t)\}$, *where* $\phi(\omega)$ *represents* $\phi(t)(\omega(t))$ *and where the expected value is taken with respect to the distribution of the random variable* ω *at time* t, *then* V *is continuous as a function of* $\phi(t)$. *Furthermore, the map* $\bar{W}$ *defined by* $\bar{W}(\phi)=\{\beta^{t}Eu(\phi(t)(\omega(t),t))\}_{t=0,1,\dots}$ *is continuous from* ψ *to* l_1.

Proof. Let $\phi^{\alpha}(t)\to\phi^{0}(t)$ in $\|\cdot\|^{1}_{\beta}$. Without loss of generality we can assume $\phi^{0}(t)=0$. Then, by the assumptions of u, if $F(t)$ represents the density function of ω at time t,

$$E(u(\phi^{\alpha}(t)(\omega(t),t))=\int_{0}^{\infty}u(\phi^{\alpha}(t)(\omega),t)F(t)(\omega)\,\mathrm{d}x.$$

$$\leq\int_{0}^{\infty}K(t)F(t)\mathrm{d}\omega<\infty, \tag{A.1}$$

where

$$K(t)=\sup_{\alpha}u(\phi^{\alpha}(t),t)\leqq u(\omega(t),t)\quad[\text{by (13)}]$$

$$\leqq u(\bar{\omega}(t),t),$$

where $\bar{\omega}(t)$ is defined in the proof of Lemma 1.

Now, since $\phi^{\alpha}(t)\to 0$ in $\|\cdot\|^{1}_{\beta}$, it follows by the definition of $\|\cdot\|^{1}_{\beta}$ that

$$\int_{0}^{\infty}\mathrm{e}^{-\beta t}(\phi^{\alpha}(t)(\omega(t),t))\,\mathrm{d}\omega\to 0\quad\text{as}\quad\alpha\to\infty. \tag{A.2}$$

Hence, by the assumptions on $F(t)$, this implies that

$$\int_{0}^{\infty}(\phi^{\alpha}(t)(\omega(t),t))F(t)(\omega)\,\mathrm{d}\omega\to 0\quad\text{as}\quad\alpha\to\infty. \tag{A.3}$$

Hence, by (A.1), (A.2) and (A.3),

$$E(u(\phi^{\alpha}(t)(\omega(t),t)))\to 0\quad\text{as}\quad\alpha\to\infty.$$

Since $u(0,t)=0$, this implies continuity of V with respect to ϕ, which shows that V satisfies indeed the Caratheodory conditions as needed in the proof of Theorem 3.

Next we show that (C.2) implies $\bar{W}$ is continuous from ψ to l_1. By condition (C.2), $\bar{W}$ maps ψ into l_1. By theorem 2.1 of Krasnosel'skii (1964, p. 22) the operator $\bar{W}$ is continuous, since it is an integral operator of the form required.

Lemma 6. *Let ψ be a subset of the space Γ with the norm $\|\cdot\|_\beta^\Gamma$. If ψ is closed coordinate-wise and condition* (13) *of section 3 is satisfied* [*i.e.*, $\omega(t) \geqq \phi(\omega(t)) \geqq 0$, $\omega(t) \geqq 0$], *then ψ is $\|\cdot\|_\beta^\Gamma$ compact.*

Proof. First note that if ψ is closed coordinate-wise, then it is $\|\cdot\|_\beta^\Gamma$ closed, since the coordinate-wise convergence topology is weaker than the $\|\cdot\|_\beta^\Gamma$ norm. Next, note that condition (13) implies that, for all t, if $\phi = \{\phi(t)\}$, the function $\phi(t)(x) \in H_\beta^1$ is bounded above in the norm, since $\phi(t)(x) \leq x$, a.e. $x \in [0, \infty)$. Hence, for any feasible ϕ in ψ,

$$\sup_{t=0,1,\dots} \|\phi(t)\|_\beta^1 \leqq \int_0^\infty e^{-\beta x} x \, dx, \tag{A.4}$$

where the right-hand side is a fixed positive real, say N. Hence, all the elements of ψ are bounded in the $\|\cdot\|_\infty^\Gamma$ norm as defined in section 3. Thus, for any initial wealth ω_0, the results of Lemma 2 of Chichilnisky (1977) apply, and hence ψ is $\|\cdot\|_\beta^\Gamma$ compact.

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