Reprinted from JOURNAL OF ECONOMIC THEORY
Vol. 15, No. 1, June 1977

Spaces of Economic Agents*

GRACIELA CHICHILNISKY

Department of Economics, Harvard University, Cambridge, Massachusetts 02138

Received July 20, 1973; revised October 4, 1976

INTRODUCTION

An economic agent is usually identified with a vector of resources and a preference relation defined on the space of commodities. In this sense, an agent is an element of the product space $X \times \mathcal{P}$, where X is a space of commodities or resources, and $\mathcal{P}$ is a space of preferences[1] defined over the space X. Proximity between agents is formalized by the introduction of a topology on the space $X \times \mathcal{P}$; given a topology X and another on $\mathcal{P}$ the space of agents inherits naturally the product of the two topologies. Spaces of commodities have standard topologies induced by those of the linear spaces in which they are imbedded, while topologies for the spaces of preferences are not in general naturally given. Topologies on spaces of preferences were studied by Kannai [6], Debreu [2], Hildenbrand [4], and Grodal [3], among others. They were introduced as a framework for the study of economies with many agents. In this context, an economy is represented by a measure defined on the σ-algebra of the Borel sets induced by the topology on the space of agents which provides a formal structure for the study of economic aggregates. Topological properties of certain economic concepts of the theory of general equilibrium and of the core are also studied in this context, for instance, the continuity of the demand correspondences [2], of the core and of the set of equilibria [5, 6] when the agent's preferences are allowed to vary.

Topologies frequently used in the literature are the Hausdorff metric introduced by Debreu [2], and the closed convergence, introduced by Hildenbrand [5]. These topologies, which have proven very useful for the theory (for instance, in the study of the continuity of the core and the equilibria) appear to be too coarse in certain cases. For example, as discussed

* This research was supported by NSF Grant GS18174. I thank Kenneth Arrow, Richard Dudley, Elon Kohlberg, and an anonymous referee for specific comments and suggestions that improved the paper.

[1] In its most general form, a preference relation θ can be thought of as a subset of $X \times X$, the product of the space of commodities with itself: $(x, y) \in \theta$ when y is preferred to x according to θ. In this sense, spaces of preferences on X can be thought of as subspaces of the space of all subsets of $X \times X$.

recently by Mount and Reiter [8], they seem to count as neighboring agents some of which are widely different in terms of certain utility indicators defined jointly in commodities and preferences, indicators used in the study of resource allocation mechanisms over a class of economic environments. The problem in these cases seems to be the lack of sensitivity of the Hausdorff metric and the closed convergence topology to the measure of the graphs of the preferences (see [8]).[2] It would thus seem desirable to count with adequate finer topologies for spaces of preferences that are more sensitive to proximity criteria according to such types of utility indicators.

It is the purpose of this paper to define a natural topology for spaces of preferences, and to explore some of its properties, including the above-mentioned sensitivity of the topology to the measure of the graphs. We proceed as follows. First an order structure $\preccurlyeq$ (a reflexive, transitive, binary relation) is given to spaces of preferences defined over a topological space of commodities (X, σ). This order $\preccurlyeq$ is shown to generate an order topology τ on a space of continuous preference relations $\mathcal{P}$, which includes preferences that are not necessarily complete, monotone, or convex.[3] The following properties are then obtained. If $X \times X$ is a normal space, the set $\mathcal{P}_1$ of irreflexive continuous preferences is a closed subset of the space of continuous preferences $(\mathcal{P}, \tau)$, and the set $\mathcal{P}_2$ of continuous transitive preferences is a closed subset of $\mathcal{P}$. An approximation result is given for agents that have preferences defined on subsets of the commodity space X: If Y is an open subset of the commodity space X, the set $\mathcal{P}^Y$ of preferences on $\mathcal{P}$ which are defined only on Y are an open subset of $\mathcal{P}$, and if Y is open and dense in X, $\mathcal{P}^Y$ is open and dense in $\mathcal{P}$. We then study the relation of the order structure $\preccurlyeq$ and the order topology τ with the Hausdorff metric (denoted τ_δ) and the closed convergence topologies (denoted τ_c).[4] We show that the order topology τ is finer than both of these topologies, when restricted to their common

[2] The definition of these type of indicators is based on the measure of the "lower contour sets" of the graphs of preferences [8]. Thus, the continuity properties of these utilities depend in part on the sensitivity of the topologies on the space of preferences to the measures of the lower contour sets of the preference's graphs. It is known that the Hausdorff metric or the closed convergence topology are not sensitive enough in this sense. One can, for instance, construct examples of sequences of closed graph preferences (θ^n) which converge to a closed graph preference θ in the Hausdorff metric or the closed convergence topology such that the measure of the graphs of the θ^n's do not converge to the measure of the graph of θ. See, for instance, the example in this paper and also [8]. These types of examples preclude the continuous extension in these topologies of the utility indicators of [8] to the class of all continuous preferences; further conditions on the preferences are required [8]. In these types of problems preferences are not necessarily monotone or convex.

[3] $\mathcal{P}$ is the space of all preferences $\theta \subset X \times X$ such that θ is an open set which is the interior of its closure.

[4] τ_c and τ_δ coincide when X is compact [5].

spaces of definition. Therefore, the results on continuity of the core and the equilibria (see [4, 6]) defined on economies or markets where the number of agents, the commodities held by them, and their preferences vary, are preserved in $(\mathcal{O}, \tau)$, as well as the continuity properties of the demand correspondences as depending on resources, preferences, and consumption sets [2].

Next we study the case where (X, σ) is given a σ-finite Borel measure μ. We show that the order topology τ is sensitive to the measure of the graphs of the preferences in a sense that τ_δ and τ_c are not: Under minimal topological restrictions on the graphs of the preferences, if a continuous preference θ is the limit of a sequence $\{\theta_n\}$ in $(\mathcal{O}, \tau)$ then the measure of the graph of θ in $X \times X$ is the limit of the measures of the graphs of the θ^n's. In [1] it is further shown that it is basically this property of τ that is needed to show continuity of utility indicators based on the measure of certain subsets of the graphs, such as those of [8]. Theorem 1 summarizes the above results. Theorem 2 investigates some topological properties of spaces of preferences with the order topology: If the space of commodities X is a compact (or compactified)[5] separable metric space then the space of continuous preferences $\mathcal{O}$ with the order topology τ is a separable metrizable locally convex ordered space, where any $\leqslant$ completely ordered connected subset which is closed and order bounded is compact. In Theorem 3 some further relations between a related order $\leqq$ and the τ_c and τ_δ topologies are also studied. The space o closed graph preferences[6] $(\bar{\mathcal{O}}, \tau_c)$ and its subspace of irreflexive, and transitive preferences $(\bar{\mathcal{O}}_3, \tau_c)$, both endowed with the order $\leqq$ are shown to be compact, normally ordered spaces. An open subbase for any compact completely ordered subset of $\bar{\mathcal{O}}$ with the topology τ_c is shown to be given by the open increasing and decreasing subsets. An extension property for continuous $\leqq$-increasing functions is given, and a τ_c continuous bounded real or vector valued function (such as a demand function) defined on a closed subset of $\bar{\mathcal{O}}$ (respectively, $\bar{\mathcal{O}}_3$) which is increasing with respect to $\leqq$ and the natural preorder of R^n $(n \geq 1)$ can be extended to a τ_c continuous, increasing function defined on all $\bar{\mathcal{O}}$ (respectively, $\bar{\mathcal{O}}_3$).

Definitions

A *preference* or *preorder* θ is a relation on X, i.e., θ is a subset of $(X \times X)$; $(x, y) \in \theta$ when y is "preferred" to x. With this definition θ is identified with its graph. A preference of preorder θ is *irreflexive* if and only if

$$(x, y) \in \theta \Rightarrow (y, x) \notin \theta.$$

[5] If X is metric separable and locally compact, one can consider its one point compactification [7], which is also a metrizable space, as done in [5] for the definition of the closed convergence topology as an extension of the Hausdorff metric to noncompact metric spaces.

[6] $\bar{\mathcal{O}}$ is a space of preferences with a closed graph in $X \times X$.

θ is *transitive* if and only if

$$(x, y) \in \theta, \qquad (y, z) \in \theta \Rightarrow (x, z) \in \theta.$$

θ is *continuous* if an only if it is an open subset of $X \times X$ such that θ is the interior of its closure, i.e., $\theta = \mathring{\bar{\theta}}$, as subsets of $X \times X$. θ is antisymmetric when $(x, y) \in \theta$ and $(y, x) \in \theta \Rightarrow x = y$. A preorder θ is an *order* when it is transitive, antisymmetric, and the diagonal D is contained in θ, where $D = \{(x, y) \in X \times X: x = y\}$. Note that the definition of continuity given here does not necessarily coincide with others given in the literature in certain cases. For instance, in [2] a continuous preference is taken to be one whose graph is closed in $X \times X$. Our definition requires an open graph, which can be thought of as the interior of the graph of a closed graph preference. These two definitions may yield different classes of preferences in some cases since, for instance, the closure of the graph of a transitive preference may not be the graph of a transitive preference, as can be seen in the following example:[7,8] Let $X = [0, 1]$. Define θ by: $(x, y) \in \theta$ if

$$x \prec \tfrac{1}{2} \quad \text{and} \quad y \prec \tfrac{1}{2}$$

or

$$x \succ \tfrac{1}{2} \quad \text{and} \quad y \prec 1;$$

θ is transitive. Now,

$$(x, y) \in \bar{\theta}$$

if

$$x \preccurlyeq \tfrac{1}{2} \quad \text{and} \quad y \preccurlyeq \tfrac{1}{2}$$

or

$$x \succcurlyeq \tfrac{1}{2} \quad \text{and} \quad y \preccurlyeq 1.$$

Thus, $(0, \frac{1}{2}) \in \bar{\theta}$, $(\frac{1}{2}, 1) \in \bar{\theta}$, but $(0, 1) \notin \bar{\theta}$. Hence, $\bar{\theta}$ is not transitive.

Let $\mathcal{O}$ be the family of all continuous preferences on a topological space (X, σ) where $X \times X$ is a connected normal space.[9]

We give an order structure to $\mathcal{O}$ by defining $\theta \prec \theta'$ if $\bar{\theta} \subset \theta'$ as subsets of

[7] This example was suggested by D. McFadden.

[8] Also, as shown by Schmeidler [10], when X is connected, if the relation θ is transitive with a closed graph and is such that the corresponding irreflexive relation θ' (given by $(x, y) \in \theta'$ if and only if $(x, y) \in \theta$ and $(y, x) \in \theta$) has an open graph, then either θ is complete, or else it is an equivalence, i.e., $(x, y) \in \theta \Rightarrow (y, x) \in \theta$.

[9] For point set topology definitions, see, for instance [7].

$X \times X$ ($\preccurlyeq$ corresponds to $\prec$ or $=$).[10] The couple $(\mathcal{O}, \preccurlyeq)$ becomes an ordered space ($\preccurlyeq$ is not a complete order on $\mathcal{O}$). The *order topology* τ on the space of preferences $\mathcal{O}$ over X is defined by giving an open subbase $\mathcal{S}$ for τ defined as follows: $U \in \mathcal{S}$ if and only if

$$U = \{\theta \colon \theta \in \mathcal{O} \text{ with } \theta \prec \theta' \text{ for some } \theta' \text{ in } \mathcal{O}\}$$

or

$$U = \{\theta \colon \theta \in \mathcal{O} \text{ with } \theta \succ \theta' \text{ for some } \theta' \text{ in } \mathcal{O}\}.^{11}$$

We next show certain desirable properties of special subspaces of preferences: the space of irreflexive continuous preferences denoted $\mathcal{O}_1$, and the space of transitive continuous preferences denoted $\mathcal{O}_2$.

Lemma 1. *The set $\mathcal{O}_1$ of irreflexive continuous preferences is a closed subset of the space of continuous preferences $\mathcal{O}$.*

Proof. Let $\theta = \lim_\alpha \theta^\alpha$, θ^α in $\mathcal{O}_1$. If θ is not irreflexive, then there exists a point $(x, y) \in \theta$ such that (y, x) is also in θ. Let U_1 be an open neighborhood of (x, y) contained in θ. Define $S(U)$, the "symmetrical set of U," by

$$S(U) = \{(v, w) \colon (w, v) \in U\}.$$

[10] One can alternatively consider the set $\tilde{\mathcal{O}}$ of closed subsets θ of $X \times X$ with $\theta = \overline{\theta^0}$; i.e., θ is the closure of its interior and $\theta^0 \neq \varnothing$. Given θ in $\mathcal{O}$, θ^0 is in $\mathcal{O}$ and is uniquely defined, and any θ in $\tilde{\mathcal{O}}$ is the closure of θ^0 in $\mathcal{O}$. Hence $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are isomorphic. In $\tilde{\mathcal{O}}$, $\theta \prec \theta^1$ can be defined by $\theta \subset \mathring{\theta}^1$. Since we here derive the topology on the space of preferences from that of the space of commodities X, for technical reasons $\mathcal{O}$ seems preferable to $\tilde{\mathcal{O}}$. The results of this paper can be translated to $(\tilde{\mathcal{O}}, \preccurlyeq)$.

[11] Note that if $\leqslant$ is the usual order in R^n given by $(x) \leqslant (y)$ if and only if $x_1 \leqslant y_1, \ldots, x_n \leqslant y_n$, then the corresponding order topology is equivalent to the usual topology on R^n. Note also that even if the order is complete (such as $\leqslant$ in R^1) it is, in general, false that the order topology (say, over R^1) restricted to some subset Y (of R^1) is equal to the order topology on Y induced by the restriction of the order on Y (cf. [7, pp. 58]). For instance, in R^1, if $Y = [0, 1] \cup (2, 3]$, then the sequence $\{X^n\} = \{2 + (1/n)\}$ converges when $n \to \infty$ to the point 1 in the order topology induced by the restriction of the order $\leqslant$ on Y. But $\{x^n\} \not\to 1$ in the usual topology of R^1, which coincides with the order topology on R^1. One must be aware of this property for the applications; see, for instance, Theorem 3. An analogous situation occurs, however, in the definition of the Hausdorff metric defined on closed sets of a compact metric space. For instance, if one considers the subspace Z of all closed graph preferences which are reflexive, then if the balls C_ϵ around θ in Z which give the ϵ-neighborhoods of θ are assumed to be in Z (i.e., to be reflexive closed sets) the topology can be shown to be, in general, different (coarser) than if one considers the ϵ-neighborhoods which are general closed sets as a basis of neighborhoods in Z, as in the usual definition of the Hausdorff metric. The point is that the Hausdorff metric on a subset of preferences Z induced by the restriction of the metric on closed sets in Z may be different than the Hausdorff metric induced by the metric between all closed sets restricted to the subset of preferences Z.

Since U_1 is a neighborhood of (x, y) in θ, $U_2 = S(U_1) \cap \theta$ is a neighborhood of (y, x) in θ. Consider the set $S(U_2)$ and let $U_3 \subset S(U_2)$ be such that $\mathring{U}_3 = U_3$, $(x, y) \in U_3$, and $\overline{S(U_3)} \subset \theta$. Let $\theta_1 \in \mathcal{P}$ be defined by $\theta_1 = U_3 \cup S(U_3)$. $\theta_1 \prec \theta$, and thus the set

$$A = \{\theta' : \theta' \succ \theta_1\}$$

is a neighborhood θ in τ. For any θ' in A there exists a (u, v) in θ' such that (v, u) is in θ' also; i.e., θ' is not irreflexive, which is a contradiction since $\theta^\alpha \to \theta$ and all the θ^α's are assumed to be irreflexive. This completes the proof.

LEMMA 2. *The subspace of all transitive continuous preferences $\mathcal{P}_2$ is closed in $\mathcal{P}$.*

Proof. Assume that $\theta = \lim_\alpha \theta^\alpha$, θ in $\mathcal{P}$, θ^α in $\mathcal{P}_2$ for all α. Then if θ is not transitive, there exists $(x, y) \in \theta$, and $(y, z) \in \theta$, with $(x, z) \notin \theta$. Let $\theta' \in \mathcal{P}$ be such that $\theta \prec \theta'$.[12] Let θ_1 be a preference in $\mathcal{P}$ such that $\theta_1 \succ \theta$ and $(x, z) \notin \theta_1$; θ_1 can be constructed as $\theta_1 = \theta' - V$, where θ' is some preference in $\mathcal{P}$ whose graph strictly contains $\bar{\theta}$, and V is a sufficiently small neighborhood of (x, z). Let θ_2 be a preference in $\mathcal{P}$ such that $\theta_2 \prec \theta$ and θ_2 contains the set $\{(x, y), (y, z)\}$. Let B be the open neighborhood of θ in $\mathcal{P}_1$ defined by

$$B = \{\theta' \in \mathcal{P} : \theta_2 \prec \theta' \prec \theta_1\}.$$

Then B is nonempty, and if $\theta^\alpha \in B$, θ^α is not transitive, contradicting the assumptions that $\theta^\alpha \to \theta$, and $\theta^\alpha \in \mathcal{P}$ for all α. This completes the proof.

Next we study the relation between topological properties of a subspace Y of the commodity space X and of preferences defined over the subspace Y. The following result yields openness of the space of agents whose preferences are defined on open subsets of the commodity space X, and openness and density of the space of agents with preferences defined over an open and dense subspace of the commodity space X.

LEMMA 3. *Let $Y \subset X$ be an open subset of X. Then the space $\mathcal{P}^Y$ of continuous preferences defined on Y is an open subset of the space of all continuous preferences $\mathcal{P}$. If Y is open and dense in X, $\mathcal{P}^Y$ is open and dense in $\mathcal{P}$.*

Proof. Since Y is open and $X \times X$ is normal, if $\theta \in \mathcal{P}^Y$ (i.e., $\theta \in \mathcal{P}$ and $\theta \subset Y \times Y$) and $Y \neq X$, then there exist θ_1 and θ_2 in $\mathcal{P}^Y$ with $\theta_1 \prec \theta \prec \theta_2$ and thus the set

$$U = \{\theta' : \theta_1 \prec \theta' \prec \theta_2\}$$

[12] If there does not exist such θ', $\theta = X \times X$, and is thus transitive.

is a neighborhood of θ contained in $\mathcal{O}^Y$. Thus $\mathcal{O}^Y$ is an open set of $\mathcal{O}$. If Y is open and dense in X, if $\theta \in \mathcal{O}^Y$ and θ is in $U_\alpha = \{\theta' : \theta' < \theta_\alpha\}$ (U_α is an element of the subbase $\mathscr{S}$ of $\mathcal{O}$), then there exists $\theta_\alpha' \subset Y \times Y$ with $\theta < \theta_\alpha' < \theta_\alpha$; θ_α' can be constructed as an element of $\mathcal{O}$ which contains $\bar{\theta}$ and an open neighborhood $V_p \subset Y \times Y$ of a point p in $Y \times Y$ such that $\bar{V}_p$ is in the interior of the set $\theta_\alpha - \bar{\theta}$. p exists by density of Y, and V_p exists by normality of X and by openness of Y in X. θ_α' is an element of $\mathcal{O}^Y$ contained in U_α. Similarly, it

$$U_\beta = \{\theta' : \theta' > \theta_\beta\},$$

there exists $\theta_\beta' \subset Y \times Y$, with

$$\theta_\beta < \theta_\beta' < \theta,$$

and θ_β' is in U_β and is an element of $\mathcal{O}^Y$. This completes the proof of density for $\mathcal{O}^Y$ in $\mathcal{O}$.

We next study the relation of this order structure $\preccurlyeq$ and the order topology τ with other topologies on spaces of preferences. If X is a compact metric space, the Hausdorff metric τ_δ is defined on the set of nonempty closed subsets of $X \times X$ (with respect to the metric d on $X \times X$, given by the product of the metric of X), by

$$\tau_\delta(E, F) = \inf\{\epsilon \in (0, \infty] : E \subset B_\epsilon(F) \text{ and } F \subset B_\epsilon(E)\},$$

where $B_\epsilon(E)$ denotes the "ϵ-neighborhood of E" defined by

$$B_\epsilon(E) = \{x \in X : d(x, E) < \epsilon\}.$$

If X is metric compact, the set of nonempty closed subsets of X endowed with τ_δ is a compact metric space [5].

If (X, d) is a locally compact separable metric space, the topology τ_c of closed convergence is defined on the set of all closed subsets F of X, by giving a subbase S_c

$$U \in S_c \quad \text{if} \quad U = \{F : F \cap K = \varnothing \text{ and } F \cap G \neq \varnothing\},$$

where K is a compact subset of X and G is an open subset of X.[13] The set of all closed subsets of $X \times X$ endowed with τ_c is a compact metrizable space [5].

The Hausdorff metric τ_δ induces a topology on the space of continuous preferences $\mathcal{O}$, given by the rule of convergence:

$$\theta^\alpha \to \theta \quad \text{iff} \quad \bar{\theta}^\alpha \xrightarrow{\tau_\delta} \bar{\theta}.$$

[13] It can be seen that $F^i \to^{\tau_c} F$ iff $\lim_i \sup_i F^i = \liminf_i F^i = F$ [5].

For simplicity, this topology on $\mathcal{O}$ is also denoted τ_δ. When X is separable and locally compact the topology of the closed convergence τ_c defines also a metric on $\mathcal{O}$, given by

$$\theta^n \to \theta \quad \text{iff} \quad \bar{\theta}^n \xrightarrow{\tau_c} \bar{\theta}.$$

These τ_δ and τ_c topologies on $\mathcal{O}$ coincide when X is compact (see [5]). Let X be a compact metric space.

LEMMA 4. *The order topology τ is finer than the Hausdorff metric τ_δ on $\mathcal{O}$.*

Proof. Let $\theta \in \mathcal{O}$. Consider an open set $U \subset X \times X$, such that $\bar{U} \subset B_{\epsilon/3}(\theta)$ and $\mathring{U} \supset \bar{\theta}$, and let $\theta_1 = \mathring{U}$. Consider an open set V in $X \times X$ such that $\mathscr{C}(\bar{V})$[14] $\subset B_{\epsilon/3}\mathscr{C}(\theta)$, and $\bar{V} \subset \theta$. Define $\mathring{V} = \theta_2$. Then $\{\theta: \theta_2 \prec \theta \prec \theta_1\}$ is an open set in τ, which is contained in $B_\epsilon(\theta)$. Thus, τ is finer than τ_δ.

Remarks. (1) The fact that τ is finer than τ_δ has some useful economic implications; among others, the continuity properties of the demand correspondence [2, 5] and those of the continuity of the core and the equilibrium [5, 6] are preserved in this topology when restricted to the adequate spaces of preferences.

(2) When X is metric separable and locally compact, one can similarly show that the order topology τ is finer than the topology of the closed convergence τ_c. In this case one considers the one point compactification $\hat{X}$ of X which is a compact metric space. The definition of the order topology now is done with respect to the compactified topology of $\hat{X}$, and since the topology of the closed convergence τ_c is defined as the Hausdorff metric on $\hat{X}$, the proof of Lemma 4 applies.

It can actually be shown that τ is strictly finer than τ_δ. Basically, the order topology τ requires that the preference of a sequence $\{\theta^n\}$ that approaches θ eventually "cover" any preference whose closure is contained in the interior of the graph of θ. An example of a sequence of preferences that converge in the Hausdorff metric sense but do not converge in the order topology is given below.

Let $X \times X$ be the unit square centered at 0 in R^2, and let D be the interior of a square of side of length $\frac{1}{2}$ centered at 0. We construct a sequence of preferences θ^n on X as follows. Let the graph of $\theta^n \in \mathcal{O}$ be the union of the set D with the interior of n strips S_j, $j = 1,..., n$ as drawn in Fig. 1, where the sets S_j are constructed so that

$$\mu\left(\bigcup_{j=1}^{n} S_j\right) < \sum_{j=1}^{n} (\epsilon/2^n), \qquad 0 < \epsilon < \mu(D).$$

[14] $\mathscr{C}(A)$ denotes the complement of the set A.

and assume, in addition, that as $n \to \infty$ the strips S_j become dense in the complement of D in $X \times X$. Then $\theta_n \to X \times X$ in the Hausdorff metric, but $\theta^n \nrightarrow (X \mathring{\times} X)$ in the order topology. These types of examples show why convergence in the Hausdorff metric does not imply convergence of the measure of the graphs.

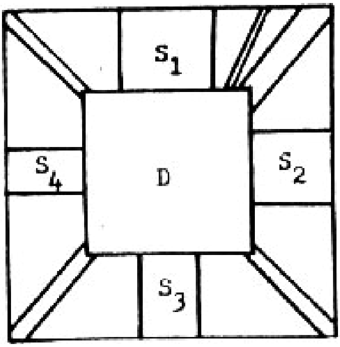

FIGURE 1

The union of the (closed) graphs of the preferences θ^n may be dense in $X \times X$, while the measure of that union may be strictly smaller than one.

We next discuss the continuity properties of the measure of the graphs of the preferences in the order topology τ. We assume now that $X \times X$ is given a σ-finite Borel measure μ, and we restrict ourselves to the subset of preferences θ in $\mathcal{O}$ with $\mu(\partial\theta) = 0$, denoted $\underline{\mathcal{O}}$.

LEMMA 5. *Let θ^α be a sequence of preferences in $\underline{\mathcal{O}}$, and let $\theta \to \theta \in \underline{\mathcal{O}}$ in the order topology τ. Then*

$$\mu(\theta) = \lim_\alpha \mu(\theta^\alpha).$$

Proof. Let V_n be a sequence of open neighborhoods of θ in $(\underline{\mathcal{O}}, \tau)$, $V_n = \{\theta' : \theta_j < \theta' < \theta_i, \theta_i, \theta_j$ in $\underline{\mathcal{O}}$ and $\theta_j < \theta < \theta_i$, with $\mu(\theta_i - \theta_j) < 1/n\}$. Then since $\theta^\alpha \to \theta$, for $\alpha > \alpha_0$, $\theta^\alpha \in V_n$ which implies that

$$\lim_\alpha \mu(\theta \triangle \theta^\alpha) = 0 \tag{1}$$

(where Δ denotes the symmetric difference). Let f and f^α be the characteristic functions of θ and θ^α, respectively. Then

$$\mu(\theta \triangle \theta^\alpha) = \int_X |f - f^\alpha| \, d\mu.$$

Let Y_n be any family of subsets of finite measure, with $X \subset \bigcup_n Y_n$. By (1)

$$\lim_\alpha \int_{Y_n} |f - f^\alpha| = 0.$$

By the Lebesgue convergence theorem

$$\int_{Y_n} \lim_\alpha |f - f^\alpha| = \lim_\alpha \int_{Y_n} |f - f^\alpha|,$$

where $\lim_\alpha |f - f^\alpha|$ denotes the pointwise limit of $|f - f^\alpha|$. It follows that $\lim_\alpha |f - f^\alpha| = 0$ a.e. in Y_n. Thus, $\lim_\alpha f^\alpha = f$ a.e. on Y_n; i.e.,

$$\lim_\alpha \mu(\theta^\alpha \cap Y_n) = \mu(\theta \cap Y_n).$$

Since this is true for each Y_n, and $X \subset \bigcup_n Y_n$, it follows that

$$\lim_\alpha \mu(\theta^\alpha) = \mu(\theta).$$

Remark 3. The results of Lemma 5 are not true when $\theta^\alpha \to \theta$ in the Hausdorff metric, as seen in the example discussed above.

In view of the results of Lemmas 1 to 5 we can now state:

THEOREM 1. *If $X \times X$ is a normal space, both the set $\mathcal{O}_1$ of irreflexive preferences and the set $\mathcal{O}_2$ of transitive preferences are closed in the space of continuous preferences $\mathcal{O}$ with the order topology τ. If $Y \subset X$ is an open subset of X, then the set $\mathcal{O}^Y$ of continuous preferences defined on Y is an open subset of $(\mathcal{O}, \tau)$, and if Y is open and dense in X, $\mathcal{O}^Y$ is open and dense in $(\mathcal{O}, \tau)$. When X is a compact metric space the order topology τ is (strictly) finer than the Hausdorff metric on $\mathcal{O}$. If X is metric separable and locally compact, and $\hat{X}$ denotes its compactification, then the order topology is (strictly) finer than the closed convergence topology on $\mathcal{O}$, when defined on continuous preferences on $\hat{X}$. If $X \times X$ is given a σ-finite Borel measure μ, and we restrict the set of continuous preferences to those with $\mu(\partial\theta) = 0$, if a net of preferences $\{\theta^\alpha\}$ converges to θ in the order topology, then*

$$\mu(\theta) = \lim_\alpha \mu(\theta^\alpha).$$

We now study some topological properties of spaces of preferences with the order topology.

An *ordered space* $(T, \leqslant)$ is a topological space equipped with an order. A subset E of $(T, \leqslant)$ is said to be *convex* whenever $a \leqslant b \leqslant c$ and a and c are in E implies b is in E. *A locally convex ordered space* is an ordered space with a convex base. A subset E of $(T, \leqslant)$ is said to be *order complete* when each nonvoid set with an upper bound has a supremum and each nonvo set with a lower bound has an infimum; E is called *linearly* or *completely* ordered if $\leqslant$ is a complete order for E (see also [9]).

THEOREM 2. *If the space of commodities X is a compact metric space, then the space of continuous preferences $(\mathcal{O}, \tau)$ is a separable and metrizable locally convex ordered space. Any linearly ordered connected subset of $(\mathcal{O}, \tau)$ which is order bounded and closed is compact. If X is a locally compact separable metric space, the same results follow for the space of continuous preferences defined on the compactification $\tilde{X}$ of X.*

Proof. First, note that $(\mathcal{O}, \tau)$ is T_1 since τ is finer than the Hausdorff metric if X is compact, and finer than the closed convergence topology if X is locally compact, by Theorem 1. This can be also seen directly as follows. Let $\theta_1 \neq \theta_2$. Without loss of generality, we can assume $\theta_1 - \theta_2 = \theta_1 \cap \mathscr{C}(\theta_2) \neq \varnothing$. This implies that $\theta_1 - \bar{\theta}_2 \neq \varnothing$. For, if $\theta_1 - \bar{\theta}_2 = \varnothing$, then $\theta_1 \subset \bar{\theta}_2 \Rightarrow \theta_1 \subset \mathring{\bar{\theta}}_2 = \theta_2$, by definition of $\mathcal{O}$, which is a contradiction. So $\theta_1 - \bar{\theta}_2$ is nonempty open set. Let V be an open ball, $\bar{V} \subset \theta_1 - \bar{\theta}_2$. Then the set

$$A = \{\theta \in \mathcal{O}: \theta > V\}$$

is a neighborhood of θ_1 in $\mathcal{O}$ which does not contain θ_2, and thus, θ_1 is not in the closure of the set $\{\theta_2\}$, which shows that $\mathcal{O}$ is T_1.

Now let $B_\epsilon(\theta)$ denote the "ϵ-neighborhood of θ in $X \times X$" given by:

$$B_\epsilon(\theta) = \{(x, y) \in X \times X \text{ with } d((x, y), \theta) < \epsilon\},$$

where d is the product metric on $X \times X$ inherited from the metric of X.

Let $\theta_1 \in \mathcal{O}$. If θ_i and θ_j are preferences in $\mathcal{O}$ with

$$\theta_i < \theta_1 < \theta_j,$$

let V_{ij} denote the set

$$\{\theta: \theta_i < \theta < \theta_j\}.$$

The sets of the form V_{ij}, for θ_i and θ_j in $\mathcal{O}$, form an open base of neighborhoods of θ_1. Let

$$\theta_{j(\epsilon)} = B_\epsilon(\theta),$$

and

$$\theta_{i(\epsilon)} = \{(x, y) \in \theta: d((x, y), \mathscr{C}(B_\epsilon(\theta))) > 2\epsilon\}$$

and let V_ϵ denote the set $\{\theta: \theta_{i(\epsilon)} < \theta < \theta_{j(\epsilon)}\}$. Note that there exists an ϵ_0

such that $\theta_{i(\epsilon)}$ is not empty for $\epsilon < \epsilon_0$, and that by compactness of $X \times X$ for any θ_i, θ_j with $\theta_i < \theta < \theta_j$ there exists an ϵ such that

$$\theta_1 < \theta_{j(\epsilon)} < \theta_j$$

and

$$\theta_1 > \theta_{i(\epsilon)} > \theta_i .$$

Thus, the V_ϵ describe a family of neighborhoods of θ_1; if the ϵ's are taken to be rational numbers, this family is countable. The space $\mathcal{O}$ is also separable; if $\theta_1 \in V_{ij}$, then by compactness of $X \times X$ there exists an $N < \infty$ such that the sequence of sets

$$\bigcup_{n=1}^{N} B_\epsilon(p_n),$$

for $p_n \in \theta_1$ and ϵ sufficiently small, approximate θ_1 as $\epsilon \to 0$ and $N \to \infty$. Since ϵ can be chosen to be rational, the θ^i's can be chosen from a countable family of finite unions of balls of rational radius with centers on a countable dense subset of $X \times X$. Therefore, by Urysohn's metrization theorem [7], $(\mathcal{O}, \tau)$ is metrizable. By the above description of a base of neighborhoods, $(\mathcal{O}, \tau)$ is also a locally convex ordered space.

Let A be any $\preccurlyeq$ linearly ordered subset of $(\mathcal{O}, \tau)$. Then by [7, pp. 162(c)] if A is closed and order bounded, A is compact if and only if it is order complete relative to $\preccurlyeq$. Since X is connected, and $X \times X$ is normal, X is order complete. Thus, since A is assumed to be connected by [7, pp. 58(c)], A is also order complete. Thus A is τ compact.

We now turn to the relationship between order and the topology τ_c. Let $\bar{\mathcal{O}}$ denote the set of all preference relations which have a closed graph. The order $\underline{\preccurlyeq}$ is defined on $\bar{\mathcal{O}}$ by $\theta_1 \underline{\preccurlyeq} \theta_2$ if $\theta_1 \subseteq \theta_2$ as subsets of $X \times X$ ($\underline{\preccurlyeq}$ is not a complete order on $\bar{\mathcal{O}}$).[16]

We need some further definitions (see also [9]): An order is called *closed* if its graphs is closed. A *compact ordered space* T is a compact topological space equipped with a closed order $\leqslant$. A subset $E \subset T$ is said to be $\leqslant$-decreasing, or *decreasing* with respect to the order $\leqslant$ when $a \leqslant b$, and $b \in E \Rightarrow a \in E$. E is called *increasing* when $a \in E$ and $b \geqslant a \Rightarrow b \in E$. An ordered space T is said to be *normally ordered* if for every two disjoint closed subsets F_0 and F_1 of Z, F_0 decreasing and F_1 increasing, there exist two disjoint open subsets A_0 and A_1 such that $A_0 \supset F_0$ and is decreasing, and $A_1 \supset F_1$ and is increasing. A function f between two ordered spaces $(T, \leqslant_1)$ and $(T, \leqslant_2)$ is called *increasing* when $x \leqslant_1 \Rightarrow f(x) \leqslant_2 f(y)$.

LEMMA 6. *$\underline{\preccurlyeq}$ is a closed order for* $(\bar{\mathcal{O}}, \tau_c)$.

[16] $\underline{\preccurlyeq}$ is strictly finer than $\preccurlyeq$, since $A \subset B$ does not imply $A \subseteq \mathring{B}$.

Proof. Consider $\{\theta^\alpha\}$ and $\{\theta^\beta\}$, with

$$\theta^\alpha \leqq \theta^\beta, \qquad \theta^\alpha \xrightarrow{\tau_c} \theta_1,$$

and

$$\theta^\beta \xrightarrow{\tau_c} \theta_2,$$

θ_1 and θ_2 in $\bar{\mathcal{O}}$. Then,

$$\theta^\alpha \subseteq \theta^\beta \Rightarrow \theta_1 = \liminf_\alpha \theta^\alpha \subseteq \limsup_\alpha \theta^\beta = \theta_2 .$$

Thus, $\theta_1 \leqq \theta_2$, which completes the proof.

COROLLARY 7. *The space of preferences $\bar{\mathcal{O}}$ with the closed convergence topology and the order $\leqq$ is a compact ordered space.*

Proof. It follows from Lemma 6, since $(\bar{\mathcal{O}}, \tau_c)$ is a compact topological space (see [3]).

Let $\bar{\mathcal{O}}_3$ denote the space of irreflexive, transitive preferences in $\bar{\mathcal{O}}$.

COROLLARY 8. $((\bar{\mathcal{O}}_3, \tau_c), \leqq)$ *is a compact ordered space.*

Proof. It follows from the fact that $\bar{\mathcal{O}}_3$ is a closed subset of $\bar{\mathcal{O}}$ with the closed convergence topology τ_c (see [3].

We can now prove the following theorem, which summarizes the relation between the order $\leqq$ and the topology τ_c on $\bar{\mathcal{O}}$.

THEOREM 3. *Let X be a locally compact separable metric space. Then:*

(1) *$(\bar{\mathcal{O}}, \tau_c)$ with the order $\leqq$ is a normally ordered space.*

(2) *For any linearly ordered closed subset F of $\bar{\mathcal{O}}$, the set consisting of all open $\leqq$-decreasing and $\leqq$-increasing subsets form an open subbase for (F, τ_c).*

(3) *Let E be a closed subset of $\bar{\mathcal{O}}$ with the following property: If $X, Y \subset E$ and $X \leqq Y$ in the space E (equipped with the topology and the order induced by $\bar{\mathcal{O}}$) then $X \leqq Y$ in $\bar{\mathcal{O}}$. Then every continuous increasing bounded real valued function defined on E can be extended to the entire space $\bar{\mathcal{O}}$ without the loss of these properties.*[16] *The same results are true for $(\bar{\mathcal{O}}_3, \tau_c)$.*

Proof. That $(\bar{\mathcal{O}}, \tau_c)$ is normally ordered follows from the fact that $(\bar{\mathcal{O}}, \tau_c)$ is Hausdorff [5] and from [9, Theorem 4, Chap. 1, Sect. 3].

[16] If one restricts oneself to a completely ordered subspace $\bar{\mathcal{O}}_1$ of $\bar{\mathcal{O}}$, the requirement that the function defined on E be bounded is not necessary for it to admit an extension to all of $\bar{\mathcal{O}}_1$ (see [9, Theorem 6, Sect. 3]).

Since (F, τ_c) is compact and linearly ordered, by [9, Theorem 5, Sec. 3], (2) follows.

Since E is closed and $\bar{\mathscr{E}}$ compact, (E, τ_c) is normally ordered also. Then (3) follows from [9, Theorem 3, Chap. 1, Sect. 2].

Remark. The result on extension of continuous $\leqslant$-increasing real valued functions can be generalized to vector valued functions, and it implies, for instance, that a continuous $\leqslant$-increasing demand function defined on a closed subset of the space of preferences such as E can actually be extended to all the space of preferences in a continuous $\leqslant$-increasing fashion.

REFERENCES

1. G. CHICHILNISKY, "Continuous Representation of Preferences," Working Paper Harvard University, 1976.
2. G. DEBREU, Neighboring economic agents, *in* "La Decision," Editions du Centre National de la Recherche Scientifique, Paris, 1969.
3. B. GRODAL, A note on the space of preference relations, *J. Math. Econ.* (1975).
4. W. HILDENBRAND, On economies with many agents, *J. Econ. Theory* **2** (1970), 161–188.
5. W. HILDENBRAND, "Core and Equilibria of a Large Economy," Princeton Univ. Press, Princeton, N.J., 1974.
6. Y. KANNAI, Continuity properties of the core of a market, *Econometrica* (1970).
7. L. KELLEY, "General Topology," Van Nostrand, Toronto/New York/London, 1955.
8. K. MOUNT AND S. REITER, Construction of a continuous utility function for a class of preferences, *J. Math. Econ.*, to appear.
9. L. NACHBIN, "Topology and Order," Van Nostrand, Toronto/New York/London, 1955.
10. D. SCHMEIDLER, A condition for the completeness of partial preference relation *Econometrica* (1971).

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium