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Existence and Characterization of Optimal Growth Paths Including Models with Non-Convexities in Utilities and Technologies

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1. INTRODUCTION

This paper is concerned with the existence and characterization of optimal growth paths in continuous time infinite horizon problems, and brings to bear on these issues mathematical techniques presented by the author in Chichilnisky (1977). These techniques, non-linear functional analysis in Hilbert (weighted L_2 and Sobolev) spaces, permit generalization, clarification and simplification of existing results. In particular, the following points are of importance:

(i) As Hilbert spaces are self-dual, continuous linear functionals on the commodity space are elements of the space, and thus have a totally natural interpretation as prices. We therefore obtain that commodity and price paths are in the same spaces, a useful property to prove existence in general equilibrium models. A failure to meet this condition has led to earlier work on characterization being troubled by the problem that supporting linear functionals are not always interpretable as prices, so that one cannot always give a price characterization of optimal and efficient paths. In the present framework this problem does not arise.

(ii) A natural way to pose the problem of finding an optimal path is as one of maximizing a continuous function on a compact set. A major advantage of the techniques we use here is demonstrated by the fact that we are actually able to follow this approach.

(iii) Because of (ii), it is possible to establish the existence of an optimal path without the use of any convexity assumptions on either the technology or the preferences. This is a major relaxation of earlier conditions, and enables existence results to be applied to models with increasing returns in production, non-convex preferences and endogenous population. For instance, in Lane (1977) both the feasible net production set and the preference ordering are in general non-convex due to the endogenous response of population growth to income. In Ryder and Heal (1973) and Wan (1970) non-convexities appear as well as in environmental and biological problems discussed for example in Clark (1976). In this latter case the logistic growth functions of biological populations introduces non-convexities in a natural way.

(iv) It can be shown that the prices that arise within this framework provide a well-defined present value for every feasible program. This makes it possible to provide a full characterization of optimal and efficient paths in terms of profit maximization without any reference to transversality conditions. This characterization is thus analogous to that used in atemporal or finite horizon problems. Such a characterization can be given globally in the convex case, and locally in the non-convex case.

Earlier work in this field (Bewley (1972), Lane (1975), Majumdar (1975), and Ryder and Heal (1973)) attempted extensions of results on existence and characterization of optimal paths of the neoclassical growth models. However, economic theory has not dealt rigorously so far with such problems in continuous time optimal growth models with non-trivial non-convexities appearing simultaneously on the technological side and on the side of the social welfare function. For example, existing sufficient conditions for existence in continuous time growth models that are based on optimal control theory require convexity assumptions on both the technology and the utilities; in addition, in those works, external conditions on the limiting behaviour of the optimal path (called transversality conditions) need to be assumed to obtain characterizations.

An extension of these earlier works to continuous time models does not seem possible. For instance, in discrete time models where the feasible production set is described by a production function f, the continuity of the function f along with norm boundedness of consumption and capital paths guarantees that the set of feasible consumption programs Y is closed in the topology of coordinate-wise convergence, and hence, closed in the Mackey topology $m(l_\infty, l_1)$, see Majumdar (1975). Along with other conditions on the technology, Y is also shown to be compact. But these arguments are no longer valid in continuous time models. The topological properties of the production set would then depend on how each instant's feasible production sets are related to the measurable structure of the space (L_∞). The consensus in the previous literature, as stated in Bewley (1972, p. 518) was that, "No general statements about this relationship seem possible". However, by the use of L_2 norms instead of coordinate-wise or Mackey topologies on L_∞ and by further arguments, here we can actually show in a continuous time case, compactness of the feasible consumption set in a norm in which additive time dependent utilities are also shown to be continuous, see Lemmas 1 and 2, and Proposition 1.[1]

2. EXISTENCE AND CHARACTERIZATION OF OPTIMAL PATHS WITH NON-CONVEXITIES

We first discuss the problem and the methodology used. One way of viewing the problem of existence of a solution in continuous time, infinite horizon models is to find adequate economic conditions under which the feasible production set is compact while, at the same time, the welfare functional to be maximized is continuous. Both these facts, compactness and continuity, must be given in the same topology. Under convexity assumptions, these problems are relatively simple, since compactness and continuity are much easier to obtain in these cases; with non-convexities the problem becomes more complex.[2] See for instance the results and discussions in Chichilnisky and Kalman (1980).

An additional problem arises in these models when one studies the characterization of solutions, from the fact that prices are dual elements of the consumption path space and the duality depends on the topology chosen. For example, dual elements of paths in L_∞ (with the sup norm) may defy economic interpretations: this produces so called "paradoxes of infinity", see for instance Chichilnisky and Kalman (1980).

The approach we take here contrasts with others in the literature in that we show that under reasonable economic assumptions Hilbert space structures can be given to both the space of feasible consumption paths and the space of capital accumulation paths The optimal paths proven to be in such spaces, are therefore proven to have an asymptotic behaviour that eliminates in particular the need for any transversality conditions. The space of consumption paths is thus given the structure of a weighted L_2 space, where the weight is related to the discount factor in intertemporal welfare.[3] Since Hilbert spaces are self-dual, this has the effect that prices are both representable by paths, and also well define a discounted present value for all paths in the space. This eliminates the problems of economic interpretation of prices which are not represented by functions. In addition, for characterization of optimal solutions in non-convex cases, one can use gradient methods which are only available in Hilbert spaces. For this, among other reasons, the spaces used here seem natural for spaces of consumption paths.

Previous work in the area has biased the choice of consumption path spaces in favour of L_∞, the space of Lebesgue measurable (a.e.) bounded functions with the sup norm. This choice has certain disadvantages produced by the lack of reflexivity of L_∞ (Bewley (1972)). However, L_∞ spaces were chosen because certain difficulties of working on L_2 (or other L_p) spaces had not been overcome before. For example, since the topology of L_2 is weaker than the sup norm of L_∞, it is more difficult to exhibit conditions on the time dependent welfare function $u(t)$ which yield appropriate continuity of the additive social welfare W to be maximized, because in general W is a nonlinear functional. Another major difficulty in the use of L_2 spaces arises from the fact that in economic theory the admissible sets of paths on which the optimization is performed are usually assumed to be contained in positive cones; for instance, admissible consumption or capital paths are positive valued through time. However, the natural positive cones of L_2 (or any L_p, $p<\infty$) spaces (i.e. the cone of positive functions), have an empty interior.[4] This produces difficulties in the application of standard tools (Hahn–Banach Theorem) used to prove existence of competitive prices for optimal or efficient programs. However, when the objective function being maximized is proven to be continuous in an L_2 topology as we do here, these difficulties can be overcome. Thus, the question of existence of characterizing prices for the optimal solution is also related to the existence of appropriate continuous functions. Because of this, in Proposition 1 continuity of the welfare function is studied, and a complete characterization is given of continuous discounted additive time dependent utilities, (not necessarily concave or quasi concave) defined on an L_2 space of consumption paths. Together with the compactness proofs of Lemmas 1 and 2, this is used to prove existence of a solution in a Hilbert space in Theorem 1.

Theorem 2 gives a global characterization of optimal paths by competitive prices for convex cases; its Corollary 1 completely characterizes efficient paths by competitive prices. Proposition 2 gives a local characterization for non-convex cases.

We now give a statement of the problem. From here on lower case letters shall represent *per capita* quantities, for instance, $k_{ji}=K_{ji}/L$, $c_i=C_i/L$, etc.; the dependence of c_i, k_{ji}, etc. on time will not always be indicated to simplify notation. In *per capita* form the model becomes

$$\max_{c_1,\ldots,c_n} W(c_1,\ldots,c_n), \tag{P}$$

where

$$W(c_1,\ldots,c_n)=\int_0^\infty e^{-\delta t}u(c_1(t),\ldots,c_n(t),t)dt,$$

and $\delta\in(0,\infty)$. The real valued Lebesgue measurable functions $c_1,\ldots,c_n$ on which the maximization is performed are restricted to a region where the following conditions (a) and (b) are satisfied, for some functions k_{ji} and l_i $(i,j=1,\ldots,n)$.[5]

(a) $$c_i+\dot{k}_i=F^i(l_i,k_{1i},\ldots,k_{ni})-\beta k_i;\qquad i=1,\ldots,n;\ \beta\in R+$$

and

(b) $$\sum_{j=1}^n k_{ji}=k_i;\qquad i=1,\ldots,n$$
$$\sum_{i=1}^n l_i=1$$
$$k_i(0)=k_{i0};\qquad i=1,\ldots,n$$

The c_i's, k_{ji}'s, and l_i's which represent *per capita* consumption, capital stocks and labour paths are all assumed to be positive real valued functions on $[0,\infty)$. Each k_i is absolutely continuous and k_i denotes its a.e. derivative.[6] The k_i's represent *per capita* rates of change of capital stocks or investment flows, and are not necessarily non negative real valued functions on $[0, \infty)$. We now define the spaces of consumption and capital paths, and other auxiliary spaces that will be useful later.

In the following we shall essentially work with three types of spaces. L_∞, a space of measurable a.e. bounded functions, a Hilbert space denoted H_λ^0 of square integrable functions with a measure given by the "discount factor" $e^{-\lambda t}$ and another Hilbert space denoted H_λ^1 of functions f in H_λ^0 such that their derivatives Df are also in H_λ^0. Formally, let $L_\infty[0, \infty)$ be the space of essentially bounded, real valued functions on $[0, \infty)$, with the sup norm denoted $\|\cdot\|_\infty$. If f and g are in $L_\infty[0, \infty)$ define the inner product:

$$(f, g)_\lambda = \int_0^\infty e^{-\lambda t} f(t) \cdot g(t) dt.$$

This inner product represents the discounted present value (at time 0) of the consumption plan f in price system g, with discount factor λ. The completion of $L_\infty[0, \infty)$ under the topology induced by the norm of this inner product $\|f\|_\lambda = (f, f)_\lambda^{1/2}$ is an L_2 space of square integrable functions on $[0, \infty)$ with density given by $e^{-\lambda t}$, i.e. $\{f \mid f$ Lebesgue measurable, $\int f^2(t)e^{-\lambda t}dt < +\infty\}$. This will be the space of consumption paths. Since L_2 spaces are self dual, any price path $g(t)$ (i.e. any continuous linear function on consumption paths) is in the same L_2 space, and, furthermore, the inner product given by $(f, g)_\lambda = \int_0^\infty e^{-\lambda t} f(t) \cdot g(t) dt$ is always well defined for any consumption path $f(t)$.

The L_2 consumption path space is denoted H_λ^0, and its L_2 norm $\|\cdot\|_\lambda$ to bring attention to the parameter λ in its definition. As the definition indicates, any path in H_λ^0 can be approximated by a sequence of L_∞ functions; this is a useful fact as this latter space is frequently used in the economic literature, e.g. Bewley (1972) and Majumdar (1975). The relation between λ and the discount factor δ of the function W of problem (P) is studied in Proposition 1 below. In view of this proposition, although all spaces H_λ^0 are equivalent L_2 spaces for any λ, only certain values of λ are adequate for this model. Similarly, one defines another space H_λ^1 consisting, as we discuss below, of continuous functions in H_λ^0 whose time derivatives are also in H_λ^0; this will be the space of capital accumulation paths.

Formally, let f and g be C_b^1 functions (continuously differentiable and bounded) and define the inner product $(f, g)_\lambda^1$ by

$$(f, g)_\lambda^1 = \int_0^\infty e^{-\lambda t} \sum_{k=0}^1 D^k f(t) D^k g(t) dt,$$

i.e.

$$(f, g)_\lambda^1 = (f, g)_\lambda + (\dot{f}, \dot{g})_\lambda,$$

and the norm $\|\cdot\|_\lambda^1$ by $\|f\|_\lambda^1 = ((f, f)_\lambda^1)^{1/2}$. The completion of C_b^1 under the norm $\|\cdot\|_\lambda^1$ is denoted H_λ^1, and it is a self dual space, a Hilbert space which is usually called a Sobolev space (see, for instance, Nirenberg (1974) and Sobolev (1963)). By a special case of Sobolev theorem $H_\lambda^1[0, \infty) \subset C^0[0, \infty)$, i.e. the functions in H_λ^1 are absolutely continuous. Self duality and the fact that its elements are absolutely continuous are two very desirable features of H_λ^1. It can be shown that H_1^λ can be also defined as

$$H_1^\lambda = \{f \mid f \text{ is absolutely continuous, } Df \in H_\lambda^0\},$$

using the Cauchy–Schwartz inequality. This definition is equivalent to the one above, and has the advantage that the only restrictions imposed by H_λ^1 bear on the square integrability of f and of Df on $[0, \infty)$ with density $e^{-\lambda t}$.[6] The previous definition helps instead to show that any functions in H_λ^1 is a limit of continuous differentiable and bounded functions, a useful feature in economics models.

We need some further notation.

Let l denote the vector $(l_1, \ldots, l_n) \in R^n$, and m the matrix:

$$\begin{pmatrix} k_{11} & \cdots & k_{n1} \\ \vdots & & \vdots \\ k_{1n} & \cdots & k_{nn} \end{pmatrix}.$$

Let k^i denote the vector $(k_{1i}, \ldots, k_{ni})$, and $k = (k_1, \ldots, k_n)$. Recall that k_i is defined to be $\sum_{j=1}^{n} k_{ji}$((b)) of Problem (P)), the sum of the coordinates of k^i.

Let $F(l, m)$ be a vector valued function $F(l, m): R^{n+} \times R^{n^2+} \to R^{n+}$, given by

$$F(l, m) = F^1(l_1, k_{11}, \ldots, k_{n1}), \ldots, F^n(l_n, k_{1n}, \ldots, k_{nn})).$$

where, for each i, $F^i(l_i, k^i): R^{(n+1)+} \to R^+$ is a function representing the technology of sector i.

We make the following assumptions on the technology of the model:

Assumption 1. F admits an extension to a real valued function F_1 defined on a neighbourhood of $R^{n+} \times R^{n^2+}$, F_1 continuously differentiable.

Assumption 2. There exists a vector $\bar{k} = (\bar{k}_1, \ldots, \bar{k}_n)$ such that

$$F^i(l_i, k^i) < \beta k_i, \qquad \text{for all } l_i \leqq 1$$

and $k_i \geqq \bar{k}_i$. Furthermore, $k_{i0} < \bar{k}_i$; for all i where k_{i0} is $k_i(0)$ as given in (b) of problem (P). This assumption is basically a technological constraint on production, it represents the fact that after certain levels of capital stocks the technology is constrained in its *per capita* increases of productivity by the costs of maintenance of (*per capita*) capital stock represented by the parameter β.

Let $C_{k_0} \times K_{k_0}$ be a set of all consumption paths in $(H_\lambda^0)^n$ and all paths of capital accumulation or allocations of capital to sectors in $(H_\lambda^1)^{n^2}$, defined by:[7]

$$(c(t), m(t)) \in C_{k_0} \times K_{k_0}$$

when

(i) $c(t) \in L_\infty[0, \infty)^{n+}$, i.e. $c(t)$ is a measurable essentially bounded non-negative path.

(ii) $m(t) \in H_\lambda^1[0, \infty)^{n^2+}$, and hence $k^i(t) \in H_\lambda^1[0, \infty)^{n+}$, for all i. Note that by Sobolev inequality Nirenberg (1974) $H_\lambda^1[0, \infty) \subset C_0[0, \infty)$, i.e. the k_{ji}'s are continuous.

(iii) $m(0)$ satisfies (b) of problem (P), i.e. $m(0) = (k_{ij}(0))$ with the corresponding $k(0)$ (also denoted $k_0) = (k_1(0), \ldots, k_n(0))$ and $k_i(0)$ satisfies (b) of (P).

(iv) Constraints (a) and (b) are satisfied a.e. for each pair $(c(t), m(t))$ in $C_{k_0} \times K_{k_0}$, for some measurable path $l(t)$ in $[L_\infty[0, \infty)]^{n+}$.

(v) $0 \leqq c(t) \leqq F(l(t), m(t))$ a.e., and

(vi) For $|h| < 1$, and all $t \in [0, \infty)$, there exists an $N > 0$ such that

$$|\Delta c(t, h)| \leqq N |\Delta F(t, h)|$$

where $\Delta c(t, h) = c(t+h) - c(t)$, and $\Delta F(t, h)$ denotes a similar variation function of F evaluated along the path, N a constant.

It can be shown that condition (v) can be replaced by the weaker condition:

$$c(t) \leqq \int_0^t F(l(t), m(t)) dt,$$

Condition (v) is a standard assumption in optimal growth models, i.e. capital cannot be consumed, see e.g. Ryder and Heal (1973). Condition (vi) bounds changes in the feasible consumption paths in small intervals of time by a constant times changes in the variation of output.

The mathematical role of condition (vi) is to rule out the case of non-existence due to optimizing sequences with unbounded oscillations, the so-called "chattering controls" which arise because non-convexities in certain environmental optimal control problems, as discussed for instance by C. W. Clark (1976, p. 171). Condition (vi) is generally not needed in convex problems, as a convex utility will assign a higher welfare value to a consumption path that exhibits a lower degree of variation (i.e. a lower derivative in absolute value) and has the same integral of consumption over time. In non-convex cases, the analogue is to require that the welfare function penalizes high variations of consumption, a condition that could replace condition (vi).

It should be noted that the problem studied here is different from the problem of optimal consumption over time of a fixed stock when no production occurs. This is because the *per capita* capital inputs of production in this model cannot themselves be consumed without their prior transformation into goods by the production technology. However, the problem of consumption of a fixed stock could be formally transformed into a similar problem to the one formalized here as

$$\max \int_0^\infty u(c,t)e^{-\delta t}dt, \quad \text{s.t.} \int_0^\infty c(t)dt \leqq S_0,$$

where S_0 is a fixed stock. In this case, if $s(t)$ is defined as $S_0 - \int_0^t c(t)dt$, so that $\dot{s}(t) = -c(t)$, condition (v) above then becomes $c(t) \leqq F(s(t))$, where $F(s(t)) = -ds(t)/dt$. Condition (vi) becomes then a restriction on the absolute value of $-ds(t)/dt$, which will not usually be needed if u is concave. For the study of the existence problem in models of optimal consumption over time of a fixed stock see, e.g. Aumann and Perles (1965) and Artstein (1974). Typically, only for restrictive families of utilities one can show that this problem has a solution. The set C_{k_0} defined by (I) to (VI) above is called the set of feasible consumption paths, and similarly K_{k_0} the set of all feasible capital matrix paths, with initial capital stock allocation k_0. The space of all feasible capital paths $k(t)$ corresponding to matrices in K_{k_0} is denoted G_{k_0}.

We also make the following assumption on the time dependent utility function: $u: R^n \times R \to R$ is said to satisfy a Caratheodory condition, if $u(c,t)$ is continuous with respect to $c \in R^n$ for almost all t in R, and is measurable with respect to t for all values of c.

The next two lemmas establish $\|\cdot\|_\lambda$ compactness under the assumptions of the sets of feasible consumption paths C_{k_0}. Together with Proposition 1 that characterizes continuity of the welfare function W, these two lemmas are used to establish existence of a solution.

Lemma 1. *Under Assumptions* 1 *and* 2 *for each initial capital stock allocation* k_0, *the set* C_{k_0} *is a* $\|\cdot\|_\infty$ *bounded* $\|\cdot\|_\lambda$ *closed subset of* $(H_\lambda^0)^{n+}$.

Lemma 2. C_{k_0} *is precompact in the* $\|\cdot\|_\lambda$ *norm.*

A proof of these two lemmas can be obtained from the arguments in Chichilnsky (1977). Define now $W(s)$ by

$$W(s) = \int_0^\infty e^{-\lambda t} u(s(t), t)dt, \qquad 1 < \lambda < 0.$$

If $u(s,t)$ satisfies the Caratheodory condition defined above, for $s \in R^n$ and $t \in [0, \infty)$, then we have the following

Proposition 1. *A necessary and sufficient condition for* W *to define a continuous function from* H_λ^0 *to* R, *is that* $|u(s,t)| \leqq a(t) + b|s|^2$ *where* b *is a positive constant and* $a(t)$

satisfies

$$\int_0^\infty a(t)e^{-\lambda t}dt < \infty, \qquad a(t) \geqq 0.$$

For a proof of this proposition, see Chichilnisky (1977).

As discussed above, the spaces H_λ^0 are equivalent for all λ. However, as seen in the statement of Proposition 1, the necessary and sufficient condition for W to be continuous in H_λ^0 involves λ as a parameter, since $a(t)$ must satisfy $\int_0^\infty a(t)e^{-\lambda t}dt < \infty$. The question arises whether the continuity of W in H_λ^0 for some λ, implies W is continuous in all other λ's, or, if not, for which range of λ's this is true. In the following we discuss this question.

Let $0 \leqq \zeta \leqq \lambda$, then

$$s_0^\alpha \xrightarrow{\|\cdot\|_\zeta} s \Rightarrow s^\alpha \xrightarrow{\|\cdot\|_\lambda} s$$

Also, $H_\lambda^0 \supset H_\zeta^0$. Therefore, if $W: H_\lambda^0 \to R$ is $\|\cdot\|_\lambda$ continuous, then $W|_{H_\zeta^0}: H_\zeta^0 \to R$ is $\|\cdot\|_\zeta$ continuous. Hence, for all $0 \leqq \zeta \leqq \lambda$, and positive valued u, if the function

$$W(s) = \int_0^\infty e^{-\lambda t}u(s(t), t)dt$$

is H_λ^0 continuous, then W is also $\|\cdot\|_\zeta$ continuous.

In view of Proposition 1 we now assume $\lambda = \delta$ where δ is the discount factor of the definition of the social welfare function W in problem (P).

Theorem 1. *Under the Assumptions* 1, 2 *and* 3, *there exists an optimal solution* c^* *to problem* (P) *in the set of feasible paths* C_{k_0}. *If* u *is strictly concave, and* F *concave,* c^* *is also unique.*

Proof. Existence follows from Lemmas 1 and 2 and Proposition 1. Uniqueness is established in a straightforward way using concavity assumptions on u and on the technology, see Chichilnisky (1977). ||

We now turn to the characterization of solution. We study the convex case first.

Definitions. $H_\lambda^{0+}[0, \infty)$ is the cone or positive paths in $H_\lambda^0[0, \infty)$, i.e. $c \in H_\lambda^{0+}[0, \infty)$ (also denoted $c \geqq 0$) when $c(t) \geqq 0$ a.e. for $t \in [0, \infty)$. $c > 0$ when $c \geqq 0$ and $c(t) > 0$ a.e. on some set of positive measure, and $c \gg 0$ when $c(t) > 0$ a.e. in $[0, \infty)$. A function $f: H_\lambda^0 \to R$ is increasing when $f(c) > f(c_1)$ if $c - c_1 > 0$. Note that if u is strictly increasing, then $W: (H_\lambda^0)^n \to R$ is an increasing function.

We next study existence of strictly positive competitive prices for c^*. First we give a definition of prices in this model. A price p is an element of the dual space of $(H_\lambda^0)^n$, a continuous linear real valued functional defined on $(H_\lambda^0)^n$ which is positive on positive elements of $(H_\lambda^0)^n$; $p(c) \in R$ is called the *value* of c in price system p. Since H_λ^0 is a Hilbert space, by definition the space of prices has the following properties: A non-zero price p must be non-zero at some period of time, i.e. $p \neq 0$ and $p \geqq 0 \to p(t) > 0$ on some set of positive measure of $R+$. A price p well defines a (finite, non-negative) value for any path of commodities c in the space, and the value of c is given by the inner product:

$$\int_0^\infty e^{-\lambda t}p(t) \cdot c(t)dt.^8$$

Theorem 2. *Under the assumptions of Theorem* 1, *when* u *is strictly increasing and both* u *and* F *are concave,* c^* *is an optimal path in* C_{k_0} *with respect to* W *if and only if there exists a* *price system p^* such that*

(i) *p^* well defines a present value for all positive consumption paths c in $(H_\lambda^0)^{n+}$ given by:*

$$p^*(c) = \int_0^\infty e^{-\lambda t} p^*(t) \cdot c(t) dt$$

(ii) $\|p^*\|_\lambda = 1$, $p^* \gg 0$, *and*

(iii) *c^* is competitive in price system p^*, (i.e. c^* maximizes the value of $p^*(c)$, for all c in C_{k_0}) and c^* minimizes expenditure (in p^*) within consumption paths yielding a welfare level of at least $W(c^*)$ (i.e. within the set $\{c \in C_{k_0}: W(c) \geqq W(c^*)\}$).*

For a proof of this theorem, see Chichilnsky (1977).

We now discuss characterization of efficient paths. We first need more notation.

Definition. A feasible consumption path c in C_{k_0} is called efficient[9] if there exists no c_1 in C_{k_0} with $c_1 > c$.

From Theorem 2 one obtains immediately, under the same assumptions:

Corollary 1. *Assume that $\bar{c}$ is an efficient path in C_{k_0}, which is also optimal with respect to a welfare function*

$$W = \int_0^\infty e^{-\lambda t} u(c(t), t) dt,$$

where u satisfies the Caratheodory conditions and is not necessarily increasing, and $W(c)$ assumes on the space of consumption paths at least one value strictly larger than $W(\bar{c})$. Then there exists a price system $\bar{p}$ such that:

(i) *$\bar{p}$ well defines a present value for all positive consumption paths c in $(H_\lambda^0)^{n+}$ given by*

$$\bar{p}(c) = \int_0^\infty e^{-\lambda t} \bar{p}(t) \cdot c(t) dt$$

(ii) $\|\bar{p}\| = 1$ *and, furthermore* $\|\bar{p}\| \gg 0$.

(iii) *If $c_1 > \bar{c}$, $\bar{p}(c_1) > \bar{p}(\bar{c})$, and*

(iv) *$\bar{c}$ maximizes the value of $\bar{p}(c)$ in C_{k_0}.*

For a proof, see Chichilnisky (1977).

This corollary is obtained from Theorem 2, pointing out that $\bar{p}$ is non zero linear functional separating C_{k_0} and a certain minimum convex set containing $A_{\bar{c}}$ and the cone $\{(H_\lambda^0)^{n+} + \bar{c}\}$.

Remark. The Hilbert space structure of H_λ^0 is important to obtain (i)—since continuity of $\bar{p} \Rightarrow \bar{p} \in (H_\lambda^0)^n$ because $(H_\lambda^0)^n$ is self dual. Note that if instead of being in $(H_\lambda^0)^n$ consumption paths would be in L_∞, continuity of $\bar{p}$ may not imply that $\bar{p}$ as a function, is non-zero. Note also that because of the Hilbert space structure of the space of consumption paths, if $\bar{c}$ is an efficient path in C_{k_0}, which maximizes in C_{k_0} the value of a price system $\bar{p}$ (as defined above), then this price must be always given as a (positive) function $\bar{p}(t)$, with the value

$$\bar{p}(t) = \int_0^\infty e^{-\lambda t} \bar{p}(t) \cdot c(t) dt$$

being well defined for all paths $c(t)$ in the space of consumption paths.

We now make further assumptions on the model in order to obtain a characterization for the optimal path without global convexity assumptions on either the function u or F.

We shall use tools of analysis on Hilbert spaces to find a positive supporting price for the optimal path in a neighbourhood of it; these conditions are obtained by gradient methods, available in Hilbert spaces but not in other infinite dimensional spaces without inner products. For the non-convex cases, we need to require more differentiability and also certain regularity conditions, which are not needed for the results in convex cases. To simplify notation we now work in a one sector case, i.e. $i = 1$.

Let the constraint (a), with initial conditions (b) of problem (P), be represented as an integral operator. This is accomplished by rewriting the differential equation

$$\dot{k} = F(l, k) - c - \beta k$$

$$k_0 = k(0),$$

as an equivalent integral equation:

$$k(t) - k(0) = \int_0^t F(l, k) - c - \beta k$$

The latter integral equation can be written as an operator equation

$$A(k, c) = 0;$$

for related integral equations see e.g. Luenberger (1969, 9.5, p. 255).

Let H_λ^2 be the Sobolev space defined as the completion of all C_b^2 (twice continuously differentiable and bounded) functions on $[0, \infty)$ with the norm $\|\cdot\|_\lambda^2$ induced by the inner product

$$(f, g)_\lambda^2 = \int_0^\infty e^{-\lambda t} \textstyle\sum_{k=0}^2 D^k f(t) \cdot D^k(t) dt,$$

i.e.

$$\|f\|_\lambda^2 = ((f, f)_\lambda^2)^{1/2}.$$

Assumption 3. We assume that the feasible consumption paths in C_{k_0} and the feasible capital accumulation paths in K_{k_0} are both contained in $H_\lambda^2[0, \infty)$. These amount to constraints on the derivatives of the consumption and capital accumulation paths which are generally satisfied by the solutions of one sector models; this is the case for example when initial capital stock equals that of the asymptotic steady state. In this case the condition is satisfied, as the constraints (v) and (vi) are never binding. Therefore one can dispense without loss of the feasible paths that do not satisfy this condition. Furthermore, we assume that the utility function $u(c, t)$ is C^1 (continuously differentiable).

Because of Sobolev's theorem (Nirenberger, 1974), Assumption 3 implies C_{k_0} is contained in C^1 and K_{k_0} is contained in C^1, i.e. the feasible consumption and capital paths are both continuously differentiable. In this case, if the constraint (a) of problem (P) is written as above by an operator A, then it follows that A is a continuously Fréchet differentiable mapping from $H_\lambda^2 \times H_\lambda^2$ into H_λ^2; see Luenberger (1969, 9.5, p. 255). In order to guarantee sufficient regularity of the solution, (continuous Fréchet differentiability) we further assume:

Assumption 4. In a neighbourhood of any optimal path c^*, the equation $A(k, c) = 0$ defines a unique implicit function $k(c)$, and $k(c)$ satisfies a Lipschitz condition

$$\|k(c) - k(v)\|_\lambda^2 \leqq R\|c - v\|_\lambda^2$$

for some positive number R.

Special cases where Assumption 4 is satisfied include the standard one sector neoclassical model, where the equation $c^* = \phi(k)$ relating capital stock with optimal consumption level gives a sufficiently well behaved function.

A related Lipschitz assumption but on technology was made by Fujita (1974), in order to prove existence and only for convex cases.

We now obtain the following characterization; its local nature makes it applicable to local decentralization only:

Proposition 2. *Under the assumptions of Theorem* 1, *Assumptions* 3 *and* 4, *for any optimal path* c^* *there exists a price system* p^* *such that*

(i) p^* *well defines a present value for all positive consumption paths in* H_λ^{0+} *given by*

$$p^*(c) = \int_0^\infty e^{-\lambda t} p^*(t)c(t)dt$$

(ii) $\|p^*\|_\lambda^2 = 1$ *and* $p^* \gg 0$ *and, in a neighbourhood* U *of* c^*,

(iii) p^* *is a local support for the feasible consumption set* C_{k_0} *at* c^*. *Furthermore, if the production function* F *is locally concave at* c^*, *then* c^* *is locally competitive in price system* p^*, *i.e.* c^* *maximizes the value of* $p^*(c)$ *for all* c *in* $U \subset C_{k_0}$.

Proof. For this proof we use a gradient argument, which is available in Hilbert spaces. Because of Assumptions 3 and 4, both the functional W and the constraints given by (a) and (b) of problem (P), and denoted as above by the operator $A(k, c) = 0$, are continuously Fréchet differentiable functions see Luenberger (1969, 9.5, p. 255 (6)). Furthermore, by Assumption 4, c^* is a regular value of the operator A, see Luenberger (1969, 9.5, p. 261). Therefore the conditions of Lemma 1, 9.3, p. 242 of Luenberger (1969) are all satisfied, and the gradient of W is orthogonal to the surface given by the constraint $A(k, c) = 0$. Note that the gradient of W at c^* exists by Assumption 3 on u. Therefore, if we define p^* to be W_{c^*}, the gradient of W at c^*, p^* gives a functional in H_λ^2, and by normalization, it can be made to satisfy $\|p^*\|_\lambda^2 = 1$. The arguments made in Theorem 2 of Chichilnisky (1977) to prove that $p^* \gg 0$ also apply to this case. This completes the proof of Part (ii).

Part (iii) is satisfied, since the gradient of W, W_{c^*} is orthogonal to the tangent of the constraint surface $A(k, c) = 0$ by Luenberger (1969, 9.3, Lemma 1). Note that the existence of this tangent surface if assured by the differentiability and regularity conditions of A (Assumptions 3 and 4).

Finally, note that Part (i) is also satisfied because $H_\lambda^2 \subset H_\lambda^0$, so that for any f in H_λ^2 and g in H_λ^0, the inner product

$$(f, g) = \int_0^\infty e^{-\lambda t} f(t) \cdot g(t)dt$$

is well defined. ||

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NOTES

1. Some of the arguments given here for Hilbert (L_2 and Sobolev) spaces could also be given for more general L_p and p-Sobolev spaces ($2 \leq p < \infty$). However, gradient arguments are available in Hilbert spaces and not in general L_p spaces; they are useful to compute solutions, and for the results of Proposition 2 here. In addition, the self-duality of Hilbert spaces makes them especially desirable for proofs of existence of general equilibria, in which fixed points of excess demand maps from the price space to itself are sought. The results of this paper are geared to provide "building blocks" for such existence proofs.

2. Another way to show existence is to prove that every maximizing sequence is compact, see also Chichilnisky and Kalman (1980). For this latter type of proof convexity (of the set and of the functions to be maximized) is extremely useful, but without convexity the theorems needed for this procedure are not available.

3. A weighted L_2 space is a space of measurable functions which are square integrable with a given (finite) measure on R^n. It is a Hilbert space with the inner product between two functions given by the integral of the product of the functions. In Chichilnisky and Kalman (1980) a similar approach was used for the study of optimal and efficient paths, in convex discrete time models.

4. An element of $L_p(R, \mu)$ is called positive if $f(t) \geqq 0$ a.e. in R. The positive cone of L_∞, $L_\infty+$ has interior points.

5. It is assumed here that the production technology is a homogeneous function so that (a) can be written in *per capita* form; the results are valid for non-homogeneous cases when the variable L of population is bounded above instead of growing at an exponential growth as assumed here. The pattern of growth of the population can be given different forms, or it can be determined endogenously. The parameter β in (a) is the sum of the depreciation rate of capital and the growth rate of the population.

6. No more general concept of derivative or of integral (e.g. as Denjoy's) could lead to a less restrictive requirement on k_i than absolute continuity, since equation (a) implies (under trivial assumption of F, i.e. $\sup\{F^i(l, k^i) \mid l \leqq 1, k_i \leqq M\} < \infty$ that $\dot{k}$ is locally bounded from above, so that its integral has to be absolutely continuous. Once the model is defined as above, it follows immediately from the measurable selection theorem (using Borel-measurability of F^i) that there is no loss of generality in assuming furthermore that also the l's and the k_{ji}'s are Lebesgue measurable.

7. $(H^0_\lambda)^n$ denotes the Cartesian product of H^0_λ with itself n times. Similarly for H^1_λ and L_∞.

8. Both these properties are, of course, not necessarily true for positive prices in L^*_∞.

9. Note that an optimal path may not be efficient, unless the welfare function W is increasing, i.e. $c' > c \rightarrow W(c') > W(c)$. Also note that for a given welfare function W, an efficient path may not be optimal with respect to W.

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