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Intergenerational Choice: A Paradox and A Solution

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Abstract. In the quest for balanced criterion for the problem of intergenerational choice we explore the role played by different groups of generations, which are represented by the Cech-Stone compactification of the naturals.

1 Introduction

In a recent paper Chichilnisky proposed two axioms that capture the idea of sustainable development [5]. The axioms require that neither the present nor the future should play a dictatorial role. From the axioms she derived a distinctive welfare criterion that leads to a new form of cost-benefit analysis. These results are summarized in the Appendix.

Here we ask whether these results can be extended to ensure equal treatment not just in the present and the future, but also other groups across time. For example, we may require equal treatment of people who live in even or in odd years, or people who are descendants of two different ethnic groups. We formalize the problem by studying orderings of associated ultrafilters of the integers, where each filter represents a group.

We show that it is generally impossible to find a criterion that ensure equal treatment to all such groups. However in some cases, where the groups are of limited diversity, the earlier results can be extended. In particular when we restrict the groups so that they evolve consistently through time, as shown in Section 5.

We take the set of events to be a countable set, identifying it with the set of naturals $\mathbf{N}$.[1] The interpretation of this set is the following: we assume that

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[1] In this paper, the setup of Chichilnisky [5] is assumed, for details see the Appendix.

it encompasses the totality of instants at which some evaluation of utility is to be exercised. A common situation where such an approach is meaningful is the classical intergenerational distribution. One interprets then the events as generations, and associates to each generation the quality of life, measured in some common units, or the amount of consumption of a certain good shared over generations. We emphasize that the events of utility evaluation are not necessarily time ordered, nor are they bound to happen. Rather, the events enumerate all possible instances at which the mentioned evaluation can happen in principle. Thus, the set can include not just the quality of life for a generation born in, say, year 1990, but *several measurements of qualities* contingent on probable scenarios of development, e.g., on whether or not of effective alternative energy production technologies in 20 years, on climate changes and such. The necessity to compare the qualities for such events is the subtlety often neglected in research on intergenerational choice, but it is clear that a responsible policy formulation should take into account different circumstances which the future generations are to face.

The purpose of this note is to examine the validity of the *evaluation procedures* used in analysis of the socio-economic development. As was shown in [**5**], many commonly used approaches are giving too much weight to some aspects of the development, neglecting others. Here we try to analyze to what extent one could attempt to refine the evaluation procedure by imposing conditions that the events contingent to some conditions are not neglected. We establish first an impossibility result that one cannot account for *all* possible threads (exact definition below), but has to restrict them somehow. A suggestion on how effectively to produce this restriction concludes the paper.

Our assumption, that the set of events is countable, is a simplification assumed just to present clearly the difficulties already arising in this simple case. We assume no additional structure on $\mathbf{N}$.

2 Notation

The background for the notation and the results of this paper is included in the Appendix, Section 7 below. Denote by $X = C_b(\mathbf{N}) = l^\infty(\mathbf{N})$ the space of feasible utilities streams, that is bounded mappings from $\mathbf{N} \to \mathbf{R}$. Let $\beta\mathbf{N}$ be the Stone-Cech compactification of $\mathbf{N}$; $\beta\mathbf{N}$ coincides as a set with the set of all ultrafilters on $\mathbf{N}$ and the ring of continuous functions on X coincides with the ring of bounded functions on $\mathbf{N}$. In other words, *any* bounded sequence indexed by elements of $\mathbf{N}$ defines a continuous function on $\beta\mathbf{N}$ and vice versa [**8**].

The space X can be given naturally the structure of Banach space with the *sup* norm on the sequences $\{u_i\} \in X$. Let now $\succ_R$ be a binary preference relation on the space of utility streams X. We assume that the relation $\succ_R$ is given by a Frechet-differentiable function $u : X \to \mathbf{R}$ defined on the Banach space of the bounded functions on $\mathbf{N}$, or, equivalently, the space $C(\beta\mathbf{N})$ (each continuous function on the compact set $C(\beta\mathbf{N})$ is bounded automatically). We will assume that the derivative is continuous in the strong topology (as a mapping from $X \to X^*, x \mapsto u'_x$), with strong (operator) norm on X^* [**7**]. The relation $\succ_R$ encodes the social preferences over the possible distribution of utilities in the very long run, which take into consideration the situations contingent to some exogenous events. The properties of $\succ_R$ are our main concern. This preference relation should exhibit the intergenerational balance for which we take as a basis the axioms of sustainability as in [**5**].

Let $x \in X$ and let l_x be the Frechet differential of u at x. By definition, l_x is an element of the dual space to X, that is a continuous linear functional on X. By Riesz' representation theorem, the element corresponds to a (unambiguously defined) Borel measure of finite variation μ_x on $\beta\mathbf{N}$ so that

$$l_x(f) = \int_{\beta\mathbf{N}} f d\mu_x.$$

We call this the measure μ_x *associated to* $\succ_R$ *at* x, or simply associated when the context makes it unambiguous.

3 Preference relation and measures on $\beta\mathbf{N}$

In what follows we restrict our attention to the derivatives of the function u defining the preference relations $\succ_R$. This simplification will allow us to concentrate on the features related to the infinite dimensionality of the problem.

First we reformulate the axioms of sustainability [5] to conform to the language of measures on $\beta\mathbf{N}$.

Sensitivity. *A preference relation* $\succ_R$ *is said to be (strongly) sensitive (at x) if the differential of u at x is strongly positive, that is if* $\xi'_i > \xi_i$ *for some* $i \in \mathbf{N}$ *and* $\xi'_j \geq \xi_j$ *for all* j, *then* $l_x(\xi') > (\xi)$.

The meaning of the strong sensitivity is that an infinitesimally small increment of the utility of a generation yields the increment of the function defining the preference relation *of the same order.*

Sensitivity of the relation $\succ_R$ implies the following property of the associated measure μ_x:

Lemma 3.1 *A preference relation* $\succ_R$ *is sensitive at x if and only if the associated measure gives positive weight to any point of* $\mathbf{N}$: $\mu_x(i) > 0$ *for any* $i \in \mathbf{N}$.

Proof: Obvious. □

Let us consider the (undesirable) properties of the preference relation which are reflected in the dominance of this or that part of the space $\beta\mathbf{N}$ with respect to the measure associated to this preference relation.

Dictatorship of the present. *A preference relation is said to be a dictatorship of the present if any preference* $x'\succ_R x$ *persists upon 'remote enough' bounded changes of the utility streams x' and x. That is, for any B there exists K such that for any pair* $x'\succ_R x$, *one has* $(x+y)\succ_R(x'+y')$, *where y, y' are arbitrary of norm at most 1 and with vanishing first K components.*

In terms of the associated measure, the dictatorship of the present is given as follows:

Lemma 3.2 *A preference relation* $\succ_R$ *is a dictatorship of the present if and only if the associated measure of the growth* $\mathbf{N}^* = \beta\mathbf{N} - \mathbf{N}$ *is zero at any point of* X.

Proof Denote by L_K the closed subspace of X of elements of $C(\mathbf{N})$ with vanishing first K components. The condition of dictatorship of the present implies that for some $D > 0$ and for any positive $c/2$ there exists number K such that $|u(x+y) - u(x)| \leq c \cdot D/2$ for all $y \in L_K$ and of norm at most D. By assumption, the function u has continuous derivative u'. It follows that for any positive $c/2$, one can find a ball of radius Δ centered at x such that the remainder in the Taylor formula of first order is at most $c/2$ times the norm of perturbation. Combining this all together, one gets (reducing, if necessary, D and Δ so that $D = \Delta$), that

$$|u'_x(\Delta \cdot y)| \leq c \cdot \Delta,$$

or $|l_x(y)| \leq c$ for all $y \in L_K, |y| \leq 1$. In particular, this is valid for the 'departing train' sequence $y_k = (0, \dots, 0, 1, 1, 1, \dots)$ with first nonzero element at k-th place $(k \geq K)$. The restriction of the function to the growth $\mathbf{N}^*$ is constant 1 and therefore, $l_x(y_k)$ converges to $\mu_x(\mathbf{N}^*)$. As the μ_x-content of the growth $\mathbf{N}^*$ is at most c with c arbitrary, we get that it vanishes.

The other direction implication is easier: if $\mu_x(y) \leq c$ for all x and all $y \in L_K, |y| \leq 1$, then the norm of $u(x+y) - u(x) = \int_0^1 u'_{x+ty}(y)dt$ can be estimated by c, yielding the desired. □

Another concept introduced in [**5**] was that of

Dictatorship of the future. *A preference relation is said to be a dictatorship of the future if any preference $x' \succ_R x$ persists for any finite change of the utility streams x' and x, that is $(x + y) \succ_R (x' + y')$ for any y, y' with only finite number of nonvanishing entries.*

Lemma 3.3 *A preference relation is a dictatorship of the future if and only if the restriction of the associated measure μ_x to the finite part $\mathbf{N} \subset \beta\mathbf{N}$ is zero at any point $x \in X$.*

Proof From the definition of the dictatorship of the future it follows that this property implies the constancy of u along the (finite-dimensional) linear subspaces of X spanned by the vectors with all but a finite number of nonzero components. This is equivalent to the vanishing of all derivatives of u with respect to coordinate vectors of X, that is to vanishing of l_x on all vectors $e_K, K \in \mathbf{N}$, or, equivalently, to $\mu_x(\{K\}) = 0$ for any $K \in \mathbf{N}$. The Lemma follows now from the countability of $\mathbf{N} \subset \beta\mathbf{N}$. □

Corollary 3.4 *A sensitive relation is not a dictatorship of the future.*

Proof Obvious. □

One can decompose any measure μ_x into its 'present' and 'future' parts:

$$\mu_x = \mu_x^P + \mu_x^F;$$

with $\mu_x^P = \mu_x \cdot \mathbf{1}_{\mathbf{N}}; \mu_x^F = \mu_x \cdot \mathbf{1}_{\mathbf{N}^*}$. The Axioms 1 and 2 (no dictatorship to present and no dictatorship to the future) read now as nontriviality of both measures μ_x^P and μ_x^F, see Theorem 7.7. in the Appendix and in [**5**].

A number of results can be formulated in terms of the interplay between the proper and improper parts of the measure μ_x, describing in particular quite different behavior of the solutions of the optimization problems on different sets of feasible utility streams when the improper part μ_x^F is dropped. We discuss this topic elsewhere.

4 Nondictatorship and threads

This section considers the non-dictatorship assumptions in more detail.

The decomposition result above states that a dictatorship of the future (present) assigns full weight to the corresponding part of $\beta\mathbf{N}$, while non-dictatorial preferences generate *mixed* measures. This conforms well with intuition.

The next step is to distinguish between different parts of the future. Let us return to the compact topological space $\beta\mathbf{N}$. The non-dictatorship conditions reformulated above are just conditions on a measure to assign positive values to both large open sets $\mathbf{N}$ and $\mathbf{N}^*$ (the latter is closed-open in $\beta\mathbf{N}$, to be precise). This ensures that the preference relation "feels" the variations of the compared utility streams.

However, these conditions alone are too rough to reflect precisely the idea of an informed preference relation accounting for possible scenarios of development. Suppose we have a further decomposition of $\beta\mathbf{N}$ into smaller (closed-open) sets. If these sets can be given a meaningful interpretation, so that the collection of the points in any of the sets can be reasonable treated in socio-economic terms, it would be reasonable to expect the associated measures of an informed preference relation to assign positive measures to these sets, that is to take into consideration the utilities for generations exposed to the events constituting the set. Let us illustrate this by an example.

Example Let $\mathbf{N}$ be interpreted just as an infinite sequence of generations, as in the standard setup of the intergenerational choice. Consider the following linear function u as defining the preference relation: if $x = (x_1, x_2, \dots, x_n, \dots)$, then

$$u(x) = \sum_{i \geq 1} a_i x_i + \lim_B x_{2i},$$

where $a_i > 0; \sum_i a_i < \infty$ and $\lim_B$ is a Banach limit [7] taken *over even indices only.* The preference relation defined by u is both future and present dictatorship free, but it exhibits the following pathology: the term responding for the evaluation of the utility streams in the very long run, *ignores completely the asymptotic behavior of the utilities over the generations with odd numbers.*

If one is troubled with such an abstract example, one can think of it in the following terms: consider a major event which influences radically the development (like the hitting of the Earth by a meteorite with mass of several hundred thousand tons, not an impossible event) and number by $2i - 1$ the generation born in year i given the catastrophe, and by $2i$ the generation of year i given no catastrophe. The functional above neglects completely the utility of the generations which happen to live if the catastrophe occurs. This preference relation, while certainly taking into account both present and future, cannot be deemed really responsible!

Of course, one can choose instead of the subset of even numbers any infinite subset of $\mathbf{N}$ such that its complement is infinite as well. One can define again the dictatorship of the present and of the future *for this particular part of* $\mathbf{N}$. It is easy to see that the dictatorships over the whole $\mathbf{N}$ and over S are logically independent if $\mathbf{N} - S$ is infinite. That means that any combination of dictatorships on S and $\mathbf{N} - S$ is possible, for example future dictatorship on S and present dictatorship on $\mathbf{N} - S$.

More formally these properties can be defined as follows:

***S*-dictatorship.** Let S be an infinite subset of $\mathbf{N}$ whose complement in $\mathbf{N}$ is infinite. Such subsets we will call *threads.* The closure of S in $\beta\mathbf{N}$ will be denoted as βS. We will say that a preference relation $\succ_R$ is S-dictatorship of the future at x, if the associated measure μ_l of S is zero, and that it is a S-dictatorship of the present if the associated measure of the growth $S^* = \beta S - S$ is zero. Notice that both of the sets S and S^* are open (and the latter is also closed) in the topology of the Cech-Stone compactification $\beta\mathbf{N}$.

5 A paradox

Now we examine the following problem: how diverse can be threads S be in which the non-dictatorship of the future or the present can be postulated? Modern political correctness suggests that we ought to consider *any* thread (i.e., infinite, coinfinite subset $S \subset \mathbf{N}$). Indeed, if a thread is omitted from consideration, it would mean that this thread is overlooked and that the individuals or generations contingent to events defined by the thread S form a discriminated group whose utilities allocation will be governed by a preference relation neglecting precisely this group. In particular, either of the dictatorships (in their S-restricted formulation) can persist on this thread.

Therefore we will call a preference relation *ideal* if it attaches a positive measure to S and to S^* for any thread S (that is, there is no S-dictatorship of the present or of the future for *any* thread S).

This is a highly desirable property. However, a difficulty arises:

Proposition 5.1 *There exists no ideal preference relation.*

Proof We build on the following fact: there exists a family of infinite coinfinite subsets S_α in $\mathbf{N}$ each two of which have just a finite intersection, which has the cardinality of continuum. A possible construction of this family is the following: identify the set $\mathbf{N}$ with the subset of rational numbers of the unit segment and choose for any number from the segment a sequence of rationals converging to it. These sequences form the family with the required properties.

The growth of the subsets from this family are open-closed subsets in $\beta\mathbf{N}$ and they do not intersect. Therefore, there exists no Borel measure assigning positive values to each of the growths. An ideal preference relation would do exactly this, as the non-dictatorship of the future implies that $\mu_x(S^*_\alpha)$ is positive for all α. Therefore it cannot exist. □

This impossibility result eliminates the hope to construct an ideal preference relation on the utilities streams. One must restrict the totality of threads S where non-dictatorship is required.

6 A solution

The failure to construct an ideal rule stems from the fact that we are trying to consider all ultrafilters, and there are too many of them. The proliferation of ultrafilters can be attributed, roughly, to two circumstances.

Firstly, there are ultrafilters which are in the closures of the threads encompassing entirely unrelated events. For example, one might distinguish a thread consisting of events at all years whose numbers (AD) are prime numbers regardless of the environmental/economic situation; certainly hardly a meaningful thread (exept, perhaps, in some kabbalist tradition).

Secondly, even given a meaningful thread, its compactification contains too many points (more precisely, as many as the whole $\beta\mathbf{N}$). If the thread describes well the development of civilization, a further refinement could be an unnecessary complication.

Therefore, a reasonable strategy to remedy the paradox of the previous section would be: (a) to restrict the variety of threads under consideration, and (b) to identify the ultrafilters in the closure of each given thread. To keep the significant features of the development accounted for, one needs, still, to have a quite involved

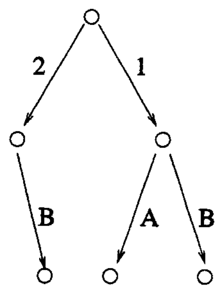

Figure 1 A caricature development tree

structure describing the possible scenarios of the development. It seems reasonable to assume that at each instant, there is just a finite number of significant circumstances influencing the utility at this instant and the course of future development. A formalization of this idea leads to the following definitions.

Recall that we identified the set of events with **N**. Now we interpret the 'events' in the extended sense, as the set of *histories leading to the events*, e.g., we consider as different the events of exhaustion in 2067 of the mineral oil resources preceded or not by the mastering of effective technologies of sun energy gaining. The set of events *with their prehistories* is again just countable, so the formalism assumed from the beginning is valid unaltered.

Now we have, however, more structure on the set of events than the time order. The events with prehistory can be provided with a *precedence order.*

Definition 6.1 *A precedence order on* **N** *is a partial order such that any interval is a finite completely ordered set.*

The intuition behind this definition is that if i precedes j then there exists a *linear* chain of events connecting i to j. In other words, *for each (hypothetical) event i there exists a unique history leading to it,* just as we assumed. A convenient way to think of such an order is to associate a tree to it (recall that a tree is a graph without cycles). One joins two nodes (elements of **N**) by an edge when one of the events immediately precedes the other (with no further points between them). We orient the edges *from* the preceding nodes (so that the directed paths in the tree correspond to future scenarios. The reverse orientation would give prehistories).

Example A caricature model of the development: in year 1, one invents a protection shield against A Huge Meteorite Hit (case 1) or not (case 2). In year 2, one has therefore two events (contingent on which of the cases realizes). Further, in year 2 a Huge Meteorite Hit happens (case A) or not (case B). In year 3 one has therefore three events, contingent on cases 1A, 1B and 2A. The corresponding tree is shown on Figure 1.

An event can give rise to several different developments; that means that one has to distinguish between the events which happen in different scenarios of the development. A model of evaluation of future development which does not distinguish between ancestors living with or without energy shortage or with or without forest devastation is hardly well fitted.

In general, possible developments form an infinite tree. The trees we consider are provided also with height function. Recall that height function is an integer-valued function on the vertices of the tree such that its values at the neighboring vertices differ by one. In our case, the time (instant of the event) plays the role of the height function. A path in a tree is descending if the height function monotonously increases along it. Two events (vertices of the tree) are joined by a descending path if and only if one of them affects the other. For a linear history case the tree is just a line.

It is worth recalling that an infinite tree which has all nodes of finite degree, has infinite length, that is it possesses an infinite oriented path (which can be unique).

For a point i, we will call all points to which i precedes, the *consequences of* i and denoted by $C(i) \subset \mathbf{N}$. Now we are ready to define our revised non-dictatorship conditions.

Definition 6.2 *A preference relation $\succ_R$ is said be neglecting consequences of i (at x) if the growth of $C(i)$ in $\beta\mathbf{N}$ (that is $\overline{C(i)} - C(i)$) has zero measure with respect to μ_x.*

For a linearly developing history, the neglect of consequences of any i is equivalent to the dictatorship of the present. It is immediate that the dictatorship of the present implies neglecting of consequences of *any* point i.

Proposition 6.3 *For any precedence order of infinite length such that the degree of each point is at most countable, there exist preference relations giving no dictatorship role to the future and not neglecting consequences of any generation i.*

Proof We just sketch the construction. Given the tree with at most countable degree of each point, one can construct a subtree having no finite leaves (that is, any oriented path there can be infinitely extended). For any vertice in this subtree one attaches positive weigths summing up to 1 to its out-edges. Set the measure of $C(i)^*$ (i.e., the growth of the set of consequences of i) to be the product of weights of the edges of the (unique) path from the root to i. This defines a (probability) measure on the σ-algebra generated by growths of $C(i)$'s assigning positive weight to each of them. Extending this measure to the Borel algebra in an arbitrary way gives the desired infinite part. Adding an appropriate finite part of the measure concludes the construction. □

A more sophisticated view of the construction above is the following: we actually *restrict the freedom of choice of threads*: no thread with causally disconnected instants are allowed now. What is more, we do not distinguish between 'interlaced' threads, that is subsets of the same infinite descending chain in the tree. Topologically, this means that first one reduces $\beta\mathbf{N}$ to a smaller set (the closure of subset corresponding to infinite descending chains) and, secondly, factorizes the resulting (still very large) compact by the 'interlacing' relation. What results is a totally disconnected Hausdorff compact with countable base (modeled by the Cantor continuum), which can be provided with a measure having reasonable properties. In more advanced terms, the resulting space is *the space of ends of the causal tree.*

Remark The naive redefinition of the non-dictatorship of the present along any descending path in the development tree, requiring a positive measure to the closure of the vertices of the path in $\beta\mathbf{N}$, is impossible: there exists continuum of paths again.

7 Appendix (by G. Chichilnisky)

See also [**5**]. Consider an infinitely lived world, an assumption that obviates the need to make decisions contingent on an unknown terminal date. Each generation is represented by an integer $g, g = 1...\infty$. Generations could overlap or not; indeed one can in principle consider a world in which some agents are infinitely long lived. In this latter case, one is concerned about the manner in which infinitely long lived agents may wish to inject considerations of sustainability into the evaluation of development paths for their own futures.

In order to compare the axioms and results to those of optimal growth theory, I shall adopt a formulation which is as close as possible to that of the neoclassical model. Each generation g has a utility function u_g for consumption of n goods, some of which could be environmental goods such as water, or soil, so that consumption vectors are in R^n, and $u_g : R^n \to R$. The availability of goods in the economy could be constrained in a number of ways, for example by a differential equation which represents the growth of the stock of a renewable resource[2], and/or the accumulation and depreciation of capital. Ignore for the moment population growth; this issue can be incorporated at little change in the results[3]. The space of all feasible consumption paths is indicated F.:1

$$F = \{x : x = \{x_g\}_{g=1,2...},\ x_g \in R^n\}. \tag{7.1}$$

In common with the neoclassical growth literature, utility across generations is assumed to be comparable. Each generation's utility functions are bounded below and above and we assume $u_g : R^n \to R^+$, and $\sup_{x \in R^n} (u_g(x)) < \infty$. This is not a restrictive assumption: one cannot have utilitities which grow indefinitely in either the positive or the negative direction when there are an infinite number of generations[4]. In order to eliminate some of the most obvious problems of comparability I normalize the utility functions u_g so that they all share a common bound, which I assume without loss to be 1:

$$\sup_g (u_g(x_g))_{x_g \in R^n} \leq 1. \tag{7.2}$$

The space of *feasible utility streams* Ω is therefore

$$\Omega = \{\alpha : \alpha = \{\alpha_g\}_{g=1,2...},\ \alpha_g = u_g(x_g))\}_{g=1,2,...} \text{ and } x = \{x_g\}_{g=1,2...} \subset F\} \tag{7.3}$$

Because I normalized utilities, each utility stream is a sequence of positive real numbers, all of which are bounded by 1. The space of all utility streams is therefore

[2]See [**3**], [**2**].

[3]Population growth and utilitarian analysis are known to make an explosive mix, which is however outside the scope of this paper.

[4]This would lead to paradoxical behavior. The argument parallels interestingly that given by Arrow [**1**] on the problem that originally gave rise to Daniel Bernouilli's famous paper on the "St. Petersburg paradox", see *Utility Boundedness Theorem*, page 27. If utilities are not bounded, one can find a utility stream for all generations with as large a welfare value as we wish, and this violates standard continuity axioms.

contained in the space of all bounded sequences of real numbers, denoted ℓ_∞[5]. The welfare criterion W should rank elements of Ω, for all possible $\Omega \subset \ell_\infty$.

7.1 Sensitivity and completeness. The welfare criterion W must be represented by an increasing real valued function on the space of all bounded utility streams[6] $W : \ell_\infty \to R^+$. The word increasing means here that if a utility stream α is obtained from another β by increasing the welfare of some generation, then W must rank α strictly higher than β [7]. This eliminates the Rawlsian criterion and the basic needs criterion, both of which are insensitive to the welfare of all generations but those with the lowest welfare. Completeness and sensitivity eliminate the Ramsey criterion as well as the overtaking criterion.

7.2 The present. How to represent the present? Intuitively, when regarding utility streams across generations, the present is the part of those streams that pertains to finitely many generations. The present will therefore be represented by all the parts of feasible utility streams which have no future: for any given utility stream α, its "present" is represented by all finite utility streams which are obtained by cutting α off after any number of generations. Formally,

Definition 7.1 *For any utility stream $\alpha \in \ell_\infty$, and any integer K, let α^K be the "$K-$cutoff" of the sequence α, the sequence whose coordinates up to and including the $K-th$ are equal to those of α, and zero after the $K-th$.*[8]

Definition 7.2 *The present consists of all feasible utility streams which have no future, i.e., it consists of the cutoffs of all utility streams.*

7.3 No dictatorial role for the present.

Definition 7.3 *We shall say that a welfare function $W : \ell_\infty \to R$ gives a dictatorial role to the present, or that W is a dictatorship of the present, if W is insensitive to the utility levels of all but a finite number of generations, i.e., W is only sensitive to the "cutoffs" of utility streams, and it disregards the utility levels of all generations from some generation on.*

Definition 7.4 *The "K-th tail" of a stream $\alpha \in \ell_\infty$, denoted α_K, is the sequence with all coordinates equal to zero up to and including the K-th, and with coordinates equal to those of α after the K-th*[9].

Formally: for every two utility streams $\alpha, \gamma \in \ell_\infty$ let (α^K, γ_K) be the sequence defined by summing up or "pasting together" the K-th cutoff of α with the K-th tail of γ. W is a dictatorship of the present if for any two utility streams α, β
$W(\alpha) > W(\beta) \Leftrightarrow$

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[5]Formally, $\Omega \subset \ell_\infty$,where $\ell_\infty = \{y : y = \{y_g\}_{g=1,\dots} : y_g \in R^+, \sup_g \mid y_g \mid < \infty\}$. Here $\mid . \mid$ denotes the absolute value of $y \in R$, which is used to endow ℓ_∞ with a standard Banach space structure, defined by the norm $\|.\|$ in ℓ_∞

$$\|y\| = \sup_{g=1,2\dots} \mid y_g \mid . \tag{7.4}$$

The space of sequences ℓ_∞ was first used in economics by G. Debreu [**6**].

[6]The representability of the order W by a real valued function can be obtained from more primitive assumptions, such as, e.g., transitivity, completeness and continuity conditions on W.

[7]Formally, if $\alpha > \beta$ than $W(\alpha) > W(\beta)$.

[8]In symbols: $\alpha^K = \{\sigma_g\}_{g=1,2\dots}$ such that $\sigma_g = \alpha_g$ if $g \leq K$, and $\sigma_g = 0$ if $g > K$.

[9]In symbols: $\sigma_K = \{\sigma_g\}_{g=1,2\dots}$ such that $\sigma_g = 0$ if $g \leq K$, and $\sigma_g = \alpha_g$ if $g > K$.

$\exists\, N = N(\alpha, \beta)$ s.t. if $K > N$, $W(\alpha^K, \gamma_K) > W(\beta^K, \sigma_K)$ for any utility streams $\gamma, \sigma \in \infty$[10].

The following axiom eliminates dictatorships of the present:

- *Axiom 1: No dictatorship of the present.*

This axiom can be seen to eliminate all forms of discounted sums of utilities, as shown in Theorem 1, Chichilnisky[11] [**5**].

7.4 The Future. For any given utility stream α, its "future" is represented by all infinite utility streams which are obtained as the "tail" resulting from modifying α to assign zero utility to any initial finite number of generations.

7.5 No dictatorial role for the future.

Definition 7.5 *Welfare function $W : \ell_\infty \to R$ gives a dictatorial role to the future, or equivalently W is a dictatorship of the future, if W is insensitive to the utility levels of any finite number of generations, or equivalently it is only sensitive to the utility levels of the "tails" of utility streams.*

Formally, for every two utility streams α, β

$$W(\alpha) > W(\beta) \Leftrightarrow$$
$$\exists\, N = N(\alpha, \beta) \text{ s.t. if } K > N,\ W(\gamma^K, \alpha_K) > W(\sigma^K, \beta_K)\ \forall\, \gamma, \sigma\ \in \ell_\infty.$$

The welfare criterion W is therefore only sensitive to the utilities of "tails" of streams, and in this sense the future always dictates the outcome independently of the present. The following axiom eliminates dictatorships of the future:

- *Axiom 2: No dictatorship of the future.*

This axiom excludes all welfare functions which are defined solely as a function of the limiting behavior of the utility streams. For example, it eliminates the lim-inf and the long run average.

Definition 7.6 *A sustainable preference is a complete sensitive preference satisfying Axioms 1 and 2.*

7.6 Existence and characterization of sustainable preferences.

Theorem 7.7 *There exists a sustainable preference $W : \ell_\infty \to R$, i.e., a preference which is sensitive and does not assign a dictatorial role to either the present or the future:*

$$W(\alpha) = \sum_{g=1}^{\infty} \lambda_g \alpha_g + \phi(\alpha), \tag{7.5}$$

where $\forall g$, $\lambda_g > 0$, $\sum_{g=1}^{\infty} \lambda_g < \infty$, and where $\phi(\alpha)$ is the function $\lim_{g\to\infty}(\alpha_g)$ extended to all of ℓ_∞ via Hahn-Banach theorem.

Proof See [**4**]. □

[10] Recall that all utility streams are in ℓ_∞ and they are normalized so that $\sup_{g=1,2\ldots}(\alpha(g)) = \|\alpha\| < 1$ and $\sup_{g=1,2\ldots}(\beta(g)) = \|\beta\| < 1$.

[11] Boundedness of the utilities is important here, although as shown above, it is not a strong assumption, see Arrow's [1] Utility Boundedness Theorem.

Theorem 7.8 *Let $W : \ell_\infty \to R^+$ be a continuous independent sustainable preference. Then W is of the form $\forall \alpha \in \ell_\infty$:*

$$W(\alpha) = \sum_{g=1}^{\infty} \lambda_g \alpha_g + \phi(\alpha) \tag{7.6}$$

where $\forall g\ \lambda_g > 0$, $\sum_{g=1}^{\infty} \lambda_g < \infty$, and ϕ is a purely finitely additive measure.

Proof See [4]. □

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