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ON FIXED POINT THEOREMS AND SOCIAL CHOICE PARADOXES

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This note explores the relationship of social choice paradoxes to fixed point theorems. A special case of the paradox proven in Chichilnisky (forthcoming), the non-existence of a continuous anonymous rule that respects anonymity with two voters and two dimensional choice spaces, is proven here to be equivalent to a fixed point problem. We indicate also how the results can be extended to higher dimensional choice spaces and any finite number of agents.

1. Introduction

The seminal works of Black (1948) and Arrow (1951) used combinatorial methods to prove the non-existence of a social choice rule that satisfies certain basic axioms. Arrow requires Pareto, non-dictatorship and independence of irrelevant alternatives conditions. Since then a large body of literature on social choice theory has developed using algebraic combinatorial methods.

More recent work by the author (forthcoming and n.d.) exhibited that topological factors gave rise to a social choice paradox: for spaces of smooth preferences defined over euclidean choice spaces, no continuous, anonymous social choice rules may exist that respect unanimity. However, these results use tools of algebraic topology, and/or index theorems of differential topology, which are not yet widely known in the mathematical economic literature. The purpose of this note is to use tools more familiar to mathematical economists to prove a special case of the paradox. We show that for two agents, and a two commodity choice space the problem of constructing a continuous anonymous aggregation rule that respects unanimity corresponds to that of the existence of a fixed point of a function on the unit disk in R^2. We also indicate how the result can be extended to k agents ($k \geqslant 2$) and n dimensional choice spaces ($n \geqslant 2$).

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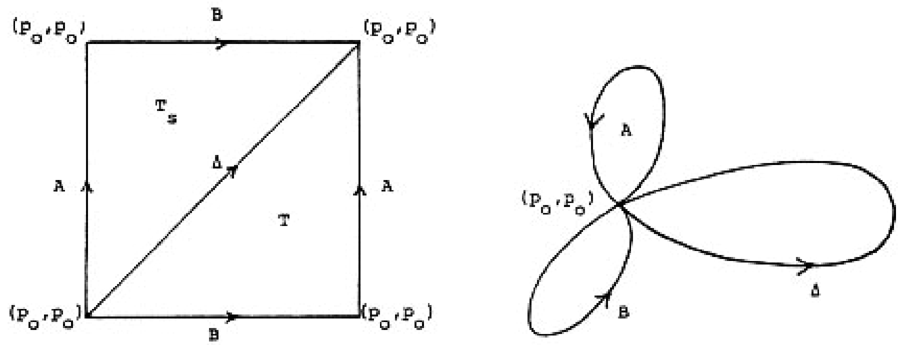

Fig. 1. $P \times P = S^1 \times S^1$ is the same space as the unit square when one does not distinguish between points on opposite sides, as in the left-hand side of the figure. A is the set $\{(p, p_0): p \in S^1\}$ and $B = \{(p_0, p): p \in S^1\}$. The boundary of the 'triangle' T is the union of the diagonal Δ with A and B, $\partial T = \Delta \cup A \cup B$; this boundary can also be represented by the three circles joined at one point, as in the right-hand side of the figure.

anonymous rule $\phi: P \times P \to P$ that respects unanimity is equivalent to the non-existence of maps from the disk into itself without fixed points (Brouwer's Fixed Point Theorem).

We shall now prove that the problem of existence of an adequate social choice rule is equivalent to the problem of extending a map from ∂D to a map on D. Let $\phi: P \times P \to P$ be a social choice rule. Note that we can *always* define on $\Delta \cup A \cup B$ a social choice rule ϕ which satisfies all the conditions, as follows: $\phi/\Delta = \text{id}_\Delta$, $\phi(p_0, p) = p_0$ $\forall p$ in S^1 and $\phi(p, p_0) = p_0$. Therefore, on the set $\Delta \cup A \cup B$ an adequate social choice rule always exists. If we could extend any social choice rule on $\Delta \cup A \cup B$ to the interior of the triangle T then, we would have constructed a social choice rule $\phi: S^1 \times S^1 \to S^1$ satisfying the desired conditions, because $S^1 \times S^1$ is the union of T and the symmetric set of T, i.e., $S^1 \times S^1 = T \cup T_s$, where $T_s = \{(x, y) : (y, x) \in T\}$, see fig. 1. Therefore, if ϕ can be continuously defined on T as an extension of the map on ∂T then since $S^1 \times S^1 = T \cup T_s$, $\phi: S^1 \times S^1 \to S^1$ could be defined satisfying all conditions. Note that T and D are homeomorphic, i.e., there exists a continuous one to one map from T onto D. Therefore, the social choice problem is equivalent to that of extending a map $g: \partial D \to \partial D$ to a map $f: D \to \partial D$. We shall now move on to the second part of the argument.

A continuous map $g: \partial D \to \partial D$ can be extended to another continuous map $f: D \to \partial D$ if and only if the map g can be deformed to a constant function mapping all of ∂D into a given x_0 in ∂D. [1] Furthermore, this result is *equivalent* to Brouwer's Fixed Point Theorem. For a proof of this equivalence see Spanier (1966). The

[1] I.e., if there exists a continuous map $F: \partial D \times [0, 1] \to \partial D$ such that $F(d, 0) = g(d)$ and $F(d, 1) = x_0 \in \partial D$ for some x_0 in ∂D, and all d in ∂D.

impossibility of extending a map from ∂D to another on D, and that the problem of extendability of maps from ∂D to D is equivalent to a fixed point theorem, it therefore follows that the social choice problem is equivalent to the problem of existence of fixed points of continuous maps from the disk to itself. This completes the proof.

Similar arguments to those of the theorem can be made (without the benefit of figures) for any n dimensional choice space, $n \geqslant 2$. This is because the Brouwer Fixed Point Theorem for the nth disk D^n is equivalent to the statement that no continuous map g: $S^{n-1} \to S^{n-1}$ can be continuously extended to another g: $D^n \to S^{n-1}$ unless g is deformable to a constant map into a point in S^{n-1}. Also, the results admit an extension to any number of agents, since arguments similar to those given in the above proof show that anonymity and unanimity prevent the extension of a continuous anonymous map of defined on $\Delta \cup S^{n-1} \cup \ldots \cup S^{n-1}$ that respects unanimity (such a map as seen above always exists) to another such map f defined on all of $S^{n-1} \times \ldots \times S^{n-1}$.

References

Arrow, K., 1951, Social choice and individual values (Yale University Press, New Haven, CT).

Black, D., 1948, On the rationale of group decision making, Journal of Political Economy.

Chichilnisky, G., forthcoming, Social choice and the topology of spaces of preferences, Advances in Mathematics.

Chichilnisky, G., n.d., Instability of social rules: A paradox without the axiom of independence of irrelevant alternatives, Discussion Paper no. 4 (Economics Workshop, Columbia University, New York).

Milnor, J., 1965, Topology from a differential view point (University Press of Virginia, Charlottesville, VA).

Spanier, E., 1966, Algebraic topology (McGraw Hill, New York).