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Actions of symmetry groups

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Abstract. This paper studies maps which are invariant under the action of the symmetry group S_k. The problem originates in social choice theory: there are k individuals each with a space of preferences X, and a social choice map $\Phi: X^k \to X$ which is *anonymous i.e.* invariant under the action of a group of symmetries. Theorem 1 proves that a full range map $\Psi: X^k \to X$ exists which is invariant under the action of S_k only if, for all $i \geq 1$, the elements of the homotopy group $\Pi_i(X)$ have orders relatively prime with k. Theorem 2 derives a similar results for actions of *subgroups* of the group S_k. Theorem 3 proves necessary and sufficient condition for a parafinite CW complex X to admit full range invariant maps *for any prime* number $k: X$ must be contractible.

1. Introduction

In a society with exactly k individuals: what conditions on the space of preferences X are needed for the existence of a continuous anonymous social choice $\Phi: X^k \to X$? The question can be formulated in simple mathematical terms: one seeks continuous maps defined from the product of a space X map to itself which are invariant under the action of a group of symmetries S_k.[1] In

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[1] S_k is the group of permutations of k elements, called the *group of permutation of k letters*. The map Φ is invariant under the group action of S_k precisely when it is symmetric in its factors, also called anonymous, see definitions below. The problem of finding continuous, anonymous social choice rules which respect unanimity was introduced and studied in [4, 5] and necessary and sufficient conditions for its solution for all k were given in [9]. Kelly discusses symmetry groups in social choice in [13].

social choice theory the space X is a space of preferences, but the question can be posed in complete generality. It is worth doing so, as this simplifies and generalizes both the question and its answer. Henceforth X is any topological space, and it could be finite or infinite dimensional.

I show that the existence maps $\Phi: X^k \to X$ which are invariant under the action of the group of symmetries S_k depends on two major features of the problem: (1) the topology of the space X, and somewhat unexpectedly, (2) the integers which are relatively prime with k. Theorem 1 establishes that full range maps $\Phi: X^k \to X$ which are invariant under the action of S_k exist only when the order of any element of X's homotopy groups is finite and relatively prime with k.[2] This result connects the group action of S_k on X with the topological properties of X and with the number theoretical properties of k. Theorem 2 extends the result to maps which are invariant under the action of subgroups of S_k.

Theorem 3 gives a general result connecting the groups of symmetries S_k with the topology of X and with properties of prime numbers: a map Φ exists for *any prime number of individuals* k, if and only if the space X is contractible. Corollary 4 shows that this extends the result of Chichilnisky and Heal [9] which proves that the contractibility of X is necessary and sufficient conditions for the existence of social choice rules which satisfy the axioms introduced in Chichilnisky [4, 5], continuity anonymity and respect of unanimity, for every number of individuals.

2. Motivation and background

The origin of the problem is as follows. Social choice theory studies the generation of social preferences from individual preferences, namely maps from the preferences of k individuals to a social preference, $\Phi: X^k \to X$ where X is a space of preferences. Φ is called a "social choice rule" and must satisfy certain ethical axioms. A natural axiom is that the outcome should not depend on the identity of the voters; this is called *anonymity* and in mathematical terms it means that the map Φ must be invariant under the action of the group of symmetries on k letters S_k. This group acts on the product space X^k by permuting the order of its factors.[3]

A special case of the problem studied here is to find maps $\Phi: X^k \to X$ which are continuous, *anonymous* and which *respect unanimity*; the latter means that when all individuals have identical preferences overall, so does society. In mathematical terms, Φ respects unanimity when it is the identity on the diagonal[4] of X^k [4].

Although these requirements are simple, the problem has no solution in general: maps Φ satisfying these axioms do not exist on unrestricted spaces of preferences [4, 5]. Baryshnikov [3] has shown recently that this social choice paradox is equivalent to Arrow's [1]. A necessary and sufficient condition for

[2] A map is full range when it covers its image at the homotopy level, see definitions below.

[3] S_k is called *the group of symmetries of k letters*: its elements are the permutations of the order of k objects.

[4] The diagonal of X^k is the space $\Delta^k = \{(x_1, \ldots, x_k) \in X^k : \forall i, j, x_i = x_j\}$.

the existence of such maps for any number of individuals k is that the preference space X be *contractible* see [9]. Therefore the obstruction to finding social choice maps lies in the topology of the space X; the problem disappears if, and only if, the global topology of X is trivial. Recently it has been discovered that a condition of "limited arbitrary" defined on trades endowments and preferences, has exactly the same role in solving this problem, Chichilnisky [8].

A natural question is whether the problem is easier to solve when there is any given, but *fixed*, number k of individuals in society.

Theorem 1 looks at this problem: it proves that the existence of a *full range*, invariant map $\Phi: X^k \to X$ places topological restrictions on X.[5] It establishes that for all $i \geq 1$, the order of any element of the homotopy group $\Pi_i(X)$ must be relatively prime with k. Maps which respect unanimity are full range, so this provides an answer to the first question posed above.

Full range maps are more general than unanimous maps, so that Theorem 1 applies also to other problems in economics where full range maps arise naturally. Example of full range maps in economics are those maps Φ respecting unanimity $\Phi/\Delta X^k = id_{\Delta X} k$, see Chichilnisky [4], as well as those maps Φ where the restriction $\Phi/\Delta X^k$ is not the identity but is *homotopic to* the identity map $id_{\Delta X} k$, which is a weaker condition than respect of unanimity, see Chichilnisky and Heal [10], and finally the case where $X = S^n$ and where the restriction map $\Phi/\Delta(S^n)^k: S^N \to S^n$ has non-zero degree, a condition which is also substantially weaker than respect of unanimity, see [10].

Theorem 2 is concerned with a more general question: given a number k and *any* subgroup G of the group S_k, what spaces X admit a map $\Phi: X^k \to X$ which is invariant under G? Theorem 2 studies the action S_j, the subgroup of S_k consisting of all permutations of j given letters within the set of k initial letters, $j < k$. Theorem 2 provides a similar charcterization of spaces X admitting maps invariant under the action of S_j, for any given $j < k$, but requiring a more stringent j-full rank condition.

From Theorems 1 and 2 I obtain Corollary 1, which shows that the sphere S^n admits no symmetric map with full range $\Psi: (S^n)^k \to S^n$ which is the identity on $\Delta(S^n)^k$, for any $k \geq 2$. This result implies James's[6] [12], Theorem 1.2 for $q = 1$. In economics this theorem applies to the aggregation of linear preferences, because the space of all linear preferences is a sphere, Chichilnisky [5]. Theorems 1 and 2 apply also to spaces of non-linear preferences which are not finite dimensional, such as those appearing in Debreu [11], and in Chichilnisky [4].[7]

[5] Theorem 1 studies maps Φ which are invariant under the action of all of S_k and for which the map induced by $\Phi/\Delta X^k$ at the homotopy level, $(\Phi/\Delta X^k)^*$, is onto: in this case Φ is called a *full range* map. The map induced by $\Phi/\Delta X^k$ at the homotopy level is denoted by $(\Phi/\Delta X^k)^*: \Pi_i(\Delta X^k) \to \Pi_i(\Delta X^k)$ for all $i \geq 1$.

[6] James [12] studies related but different symmetric maps $\Phi: (S^n)^k \to S^n$: he says that a map Φ is of type q when degree $|\Phi/i(S^n)|: S^n \to S^n = q$, where $i(S^n) = \{(x, e, \ldots, e\}$, $\forall x \in S^n, e$ a fixed element in $S^n\}$. When $X = S^n$, his conditions on the type of Φ are related to but different from ours, which are defined on the restriction of Φ on $\Delta(S^n)^k$. However, his conditions make contact with ours in certain cases: our Theorems 1 and 2 imply his Theorem 1.2 for the case where $X = S^n$ and the map Φ is of James' type $q = 1$, see Corollary 1.

[7] Spaces of preferences are often infinite dimensional, see [4].

Finally, Theorem 3 provides a necessary and sufficient condition on a parafinite CW complex X to admit a full range map $\Phi: X^p \to X$ which is invariant under the action of S_p *for every prime number* $p > 1$. Corollary 3 shows that such an invariant map Φ exists for every prime p, if and only if it exists for every integer $k \geq 1$, and this is if and only if X is contractible, a result which extends the necessary and sufficient conditions of Chichilnisky and Heal [9], which are valid for any number of individuals.

3. Definitions and notation

A social choice map is[8] a continuous map $\Phi: X^k \to X$ which is anonymous and respects unanimity, as defined below [4]. The space X represents a space of preferences; such spaces come in many forms. In this paper I consider general topological spaces X. The only requirement on X is that it should have finitely generated homotopy groups; this includes manifolds, polyhedra, simplicial complexes, etc. X could be finite or infinite dimensional.

Several "models" exist for the space of preferences X. A standard description of a preference ordering [4] is as follows. An individual's preference for choice in R^n is defined by a unit vector field indicating at each point the "most preferred direction" of increase, which is (locally) the gradient of a utility function, a real valued function on R^n. In mathematical terms, therefore, a preference is a codimension-one oriented foliation of R^n.[9] Special cases of preferences are as follows. Consider the space P of all linear preferences[10] defined on choices in euclidean space R^n. In this case $P = S^{n-1}$ then $n-1$-th sphere. To see that this is the space of linear preferences on R^n, note that in the linear case, each linear preference is uniquely represented by a vector of unit length, see [4, 11]. P is therefore the $n-1$-dimensional sphere S^{n-1}, perhaps plus the point $\{0\}$ if the "total indifference" is allowed [5]. The space of linear preferences fits well my requirements because all the homotopy groups of spheres are finitely generated [15, Corollary 16 p. 509]. Note that when preference are not linear, the space of preferences X may not be finite dimensional; in some cases it is a parafinite CW complex[11] [6]. For Theorem 3, I also require that the space of preferences be contained in some linear space, finite or infinite dimensional.

Let S_k denote the group of all permutations of k elements, which is called usually the group of permutation of k letters. This group acts on the product space X^k by permuting its factors, as follows: if $\sigma \in S_k$, $\sigma: X^k \to X$ by $\sigma(p_1, \dots, p_k) = (p_{\sigma(1)}, \dots, p_{\sigma(k)})$.

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[8] Another version was introduced earlier by Arrow [1], who was concerned with a finite set of choices, and with the space P of all orderings on that space of choices, so that P is also a finite set. Baryshnikov [3] established recently the equivalence of the two versions of the social choice paradox.

[9] The space of smooth preferences P is shown to be the space of all C^1 oriented codimension-one foliations of R^n [11, 4, 6]. It is infinite dimensional.

[10] A *linear preference* is an ordering on R^n induced by a linear real valued function on R^n.

[11] Parafinite CW complexes fit my requirements because they have finitely generated homotopy groups, see [15].

The permutations of $j \leq k$ letters, say the letters $1, \ldots, j$, defines a subgroup of S_k, denoted S_j. Let $X^j = \{(p_1, \ldots, p_j, x, \ldots, x)\}$, for a given base point $x \in X$, $p_i \in X\}$. Then $X^j \subset X^k$, and the subgroup $S_j \subset S_k$ acts on X^k as follows: $\forall \sigma \in S_j$,

$$\sigma(p_1, \ldots, p_j, \ldots, p_k) = (p_{\sigma(1)}, \ldots, p_{\sigma(j)}, p_{j+1}, \ldots, p_k).$$

More general embeddings can also be considered.

Definition 1. Φ is invariant under the action of S_k when $\Phi = \Phi \circ \sigma$ for all σ in S_k, where "$\circ$" indicates composition. Similarly define invariance of Φ under the action of the subgroup S_j.

Continuity of the map Φ is required for practical reasons: so that Φ can be approximated by statistical sampling or "polling".

Definition 2. Φ is called anonymous when it is invariant under the action of the symmetry group S_k.

Definition 3. A map $\Phi: X^k \to X$ which is the identity on the diagonal of X, i.e. $\Phi/\Delta X = id_{\Delta X}$, $\Delta X = \{(p_1, \ldots, p_k)$, s.t. $\forall i, j, p_i = p_j\}$, is said to respect unanimity [4].

This condition is weakened to a condition called "full range":

Definition 4. Φ has full range when $\forall i \geq 1$, $(\Phi/\Delta X^k)^*: (\Pi_i(X)^k) \to \Pi_i(X)$ is onto.

Definition 5. Φ has j-full range when $\forall i \geq 1$, $(\Phi/\Delta X^k)^*: (\Pi_i(X)^j) \to \Pi_i(X)$ is onto for some $j \leq k$, all $n \geq 1$, and for a standard embedding of ΔX^j into X^k.

Full-range can be interpreted as the ability of j individuals who act unanimously to control the outcomes, Chichilnisky [7]; when $X = S^n$ it is implied by the condition that degree $(\Phi/\Delta X^k)^*: (S^n)^k \to S^n \neq 0$, the intended interpretation of which is that by varying sufficiently the preferences which occur with unanimity, all possible social preferences can be achieved. For applications, see [10].[12]

The purpose of this article is to provide, for given k, a characterization of spaces X admitting maps $\Phi: X^k \to X$ which are invariant under the action of S_k or its subgroups; related examples appear in the work[13] of James [12] who studied symmetric maps defined on products of spheres, i.e. where $X = S^n$.

Definition 6. Two integers a and b are called *relatively prime* when their only common factor is the number 1.

4. Results

Theorem 1. *Let $\Phi: X^k \to X$ be a continuous full range map which is invariant under the action of S_k on X^k. Then for all $i \geq 1$, the homotopy group $\Pi_i(X)$*

[12] The condition admits also an interpretation in terms of the manipulability and the control of social preferences, see [7, 9].

[13] James [12] studies symmetric but otherwise different maps, which are defined exclusively on spheres, $f: (S^n)^k \to S^n$, and are of "type q". A map is said to be of type q when $q = \text{degree } [f/i(S^n)]: S^n \to S^n$, where $i: S^n \to (S^n)^k$ is defined by $i(x) = (x, e, \ldots, e)$ for all $x \in S^n$, e a fixed element in S^n. Although his conditions and spaces are generally different from mine, they are special cases of mine in some cases, and in particular, my Theorems 1 and 2 imply his Theorem 1.2 where the space $X = S^n$ and the maps are of type $q = 1$, see Corollary 1 below.

has no free part, and the order of any element of its torsion is relatively prime with k.

Proof. For each $i \geq 1$ consider the map Φ^* induced by Φ at the i-th homotopy level:

$$\Phi^*: \Pi_i(X^k) \to \Pi_i(X).$$

By the assumptions, $\Pi_i(X)$ is abelian; this is because $\Pi_i(X)$ is always abelian when $i \geq 2$ (see [15]) and, for $i = 1$, $\Pi_1(X)$ is abelian when there exists a map $\Phi: X^k \to X$ satisfying $\Phi/\Delta X^k = id_{\Delta X^k}$ and invariant under the action of S_k, see [9, Theorem 1].

By the assumptions on X, $\Pi_i(X)$ is a finitely generated group. Thus $\Pi_i(X)$ is the direct sum of a finite number of infinite cyclical groups each isomorphic to the integers, denoted Z, and of cyclical groups of finite order, these latter denoted Z_{q_i}, where q_i is a power of a prime number [14]. The direct sum of the infinite cyclical groups is the free part of $\Pi_i(X)$; the direct sum of the rest is its torsion defined as $T(\Pi_i(X)) = \{x \in \Pi_i(X): mx = 0 \text{ for some integer } m \geq 1\}$. Note also that $\Pi_i(X^k)$ is isomorphic to the product of k copies of $\Pi_i(X)$, i.e. $\Pi_i(X^k) = \prod \Pi_i X$ for all $i \geq 0$ [15, p. 419, No. 5 of B].

Now let y be a generator of the free part of $\Pi_i(X)$. Because Φ has full range, there exists $x \in \Pi_i(X)$ such that

$$\Phi^*(x, \ldots, x) = y. \tag{1}$$

Also, if e is the identity element in the group $\Pi_i(X)$, since Φ^* is a group homomorphism,

$$\Phi^*(x, \ldots, x) = \Phi^*(x, e, \ldots, e) \# \Phi^*(e, x, \ldots, e) \# \cdots \# \Phi^*(e, \ldots, x), \tag{2}$$

where $\#$ is the group operation in $\Pi_i(X)$, because $(x, \ldots, x) = (x, e, \ldots, e) \# \cdots \# (e, \ldots, x)$. Note that all factors in the right hand side of (2) are the same because of symmetry, say equal to $\Phi^*(x, \ldots, e) = z$. The invariance with respect to S_k, (1) and (2) imply

$$y = k(\Phi^*(x, e \ldots, e) = kz \quad \text{for some } z \in \Pi_i(X), \tag{3}$$

because $\Pi_i(X)$ is abelian. Since y is a generator in the free part of $\Pi_i(X)$, $y = kz$ implies that y must be zero. Thus by (1) $\Pi_i(X)$ has no free part. Similarly, if y is a generator of the torsion of $\Pi_i(X)$, then (1) (2) and (3) imply that y is divisible by k. I now show that, since y generates the group Z_{q_i}, then q_i must be relatively prime with k, i.e. there exists integers a and b such that $aq_i + bk = 1$. Suppose k and q_i are relatively prime. Then there exist a and b with $aq_i + bk = 1$, so that $ybk = y - ayq_i = y$, because $ayq_i = 0$ by construction. There exists therefore an element $z = yb$ with $zk = y$, as required by (3). Reciprocally, if there is a z with $kz = y$, then $zk + q_i y = y$ (because $q_i y = 0$) so that $y^{-1}zk + q_i = 1$, and k and q_i are relative prime. □

The following theorem considers a more general situation, where *any* subgroup of S_k acts on X^k.

Theorem 2. *Let $\Phi: X^k \to X$ be continuous, invariant under the action of the subgroup S_j of S_k, $j < k$, and having j-full range. Then $\Pi_i(X)$ has no free part and the order of any elements of its torsion is relatively prime with k.*

Proof. The proof follows that of Theorem 1. If Φ has j-full range, then for all $y \in \Pi_i(X)$ there exists an x in $\Pi_i(X)$ such that

$$\Phi^*(x, \ldots, x, e, \ldots, e) = y,$$

where the number of x's in the argument is j. Now

$$\begin{aligned}\Phi^*(x, \ldots, x, e, \ldots, e) &= \Phi^*(x, e, \ldots, e) \# \Phi^*(e, x, e, \ldots, e) \# \cdots \# \Phi^*(e, \ldots, x, \ldots, e)\\ &= jz,\end{aligned}$$

$z = \Phi^*(x, \ldots, e)$ because Φ is invariant under the action of S_j. Choosing y to be a generator of the free part of $\Pi_i(X)$, this implies $y = 0$, so that $\Pi_i(X)$ has no free part. If y is a generator of the torsion of $\Pi_i(X)$, y is divisible by j so that as shown in Theorem 1, the order of y must be relatively prime with j. □

For the following result, recall that S^n is the n-dimensional sphere in R^{N+1}:

Corollary 1. *For any $n \geq 1$, $k \geq 2$ and $j \leq k$, there exists no continuous full range map $\Phi:(S^n)^k \to S^n$ which is invariant under the action of the group S_k. In particular there is no continuous anonymous map $\Phi:(S^n)^k \to S^n$ which respects unanimity.*

Proof. This follows from Theorems 1 and 2 noting that $\Pi_n(S^n) = Z$, the free group in one generator [15]. Note that a map which respects unanimity has a j-full rank for $j = k$. □

This proves Theorem 1.2 of James [12] for maps of type $q = 1$.

In the following theorem X is a parafinite CW complex contained in a linear space L.

Theorem 3. *There exists a continuous full range $\Phi: X^p \to X$ invariant under the action of the group S_k for every prime number p if and only if X is contractible.*

Proof. Assume Φ exists. Then for all $i \geq 1$, $\Pi_i(X)$ must be zero, because by Theorems 1 and 2 it can have no free part and no torsion. Since by assumption X is a parafinite CW complex, X must then be contractible. The converse is established as follows. Let $C(X)$ be the convex hull of X. Since $\Pi_i(X) = 0$ for all $i \geq 1$, then there exists no obstruction to extending the inclusion map. $i: X \to C(X)$ to all of $C(X)$, so that X is a retract of $C(X)$, i.e. there exists $r: C(X) \to X$ s.t. $r/X = id/X$. The convex addition $\lambda_p(x_1, \ldots, x_p) = (1/p)\sum_i x_i$ defines a map $\lambda_p: (C(X))^p \to C(X)$ for all p, which is invariant under the action of S_p and which is the identity on the diagonal $\Delta(C(X))^p$. The composition map[14] $\Phi = i^p \circ \lambda_p \circ r: X^p \to X$ is invariant for every prime integer p, and satisfies $\Phi/\Delta X^p = id_{\Delta X^p}$, which implies that Φ has a full range. □

Theorem 4. *There exists a continuous full range map $\Phi: X^k \to X$ invariant under the action of S_k for every number $k \geq 2$, if and only if it exists such a map for every prime number $p \geq 2$.*

Proof. This follows from Theorem 3 and from Theorem 1 of Chichilnisky and Heal [9] which establishes that a continuous anonymous social choice rule exists respecting unanimity for any number of individuals if and only if X is contractible. □

[14] i^p is the inclusion map $i^p: X^p \to C(X)^p$.

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